ON NORMAL NUMBERS

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§1. *Introduction*

Let $F(m, R, b, x)$ denote the number of times that the digit b occurs in the first m places when the fractional part of x is expressed in the scale of R. If $F(m, R, b, x)/m \to 1/R$ for every $b < R$, then x is said to be *simply normal* in the scale of R.

Definition A: If x is simply normal in all the scales, $r, r^2, r^3, \ldots$, then x is *normal* in the scale of r.

Definition B: If all the numbers $x, rx, r^2x, \ldots$ are simply normal in all the scales $r, r^2, r^3, \ldots$, then x is *normal* in the scale of r.

In a sample page of the '*Mathematics Review*' it was pointed out that my proof[1] of a theorem of Champernowne's was inadequate. Further, Dr. Vijayaraghavan, by giving an example[2], suggested that my remark, about definition B, that it seemed to contain superfluous conditions,[2] might not be correct.

This paper owes its origin to the above remarks. In this paper, I make the proof adequate and prove that the two definitions are equivalent. So we may take the simpler definition A.

Further Notations and Symbols

Unless otherwise mentioned, the scale of notation is 10.

Let $x+1, x+2, \ldots, x+q$ be q consecutive integers of l digits, and X stand for any of these numbers. $K = \cdot a_1 a_2 \ldots a_{ql}$ is the number that we get when we write down $x+1, \ldots, x+q$ one after another in the same

[1] *Proc. Ind. Acad. Sci.*, 1939, **10**, No. 1.

[2] If $a_1 = 22$, $a_2 = 12$, $a_3 = 13$, and $a_4, a_5, \ldots, a_{100}$ are other 97 numbers less than 100, and $x = \cdot \overset{*}{a}_1 a_2 \ldots \overset{*}{a}_{100}$, then $10x = 2 \cdot \overset{*}{b}_1 b_2 \ldots \overset{*}{b}_{100}$, where $b_1 = 21$, $b_2 = 21, \ldots$ Since 21 occurs twice, $10x$ is not simply normal, but x is simply normal in the scale of 100. It is understood that if $a_r = 0, 1, \cdots$, or 9, then it should be written as 00, 01, $\cdots$, or 09.