ON NORMAL NUMBERS

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§ I

LET $m(b)$ denote the number of times that the digit b occurs in the first m places when x is expressed in the scale of r. If $m(b)/m \to 1/r$ for every $b < r$, then x is said to be *simply normal* in the scale of r. If x is simply normal in all of the scales $r, r^2, r^3, \cdots$ then x is said to be *normal* in the scale of r.[1] The number $\cdot 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12 \cdots$ formed by writing down all the positive integers in order, in decimal notation, is proved to be normal by Champernowne.[2] Hardy and Wright say 'The proof that this is so is more troublesome than might be expected'. The object of this note is to give an easy proof for this result and consider some generalisations.

§ II

Unless otherwise mentioned, in this section, the numbers are assumed to be expressed in the scale of r; and 'numbers' means positive integers.

F (N) denotes the total number of digits in all the numbers $\leqslant$ N; K (N) denotes the total number of times that the digit b occurs in all the numbers $\leqslant$ N, where $b < r$; and $K_t(N)$ denotes the total number of numbers which contain b in the tth place (from the right) and which do not exceed N.

$$n = [\log N/\log r].$$

Lemma 1.—$F(N) = n \cdot N + O(N)$.

If $r^{t-1} \leqslant x < r^t$, x contains t digits. So

$$\begin{aligned} F(N) &= F(N) - F(r^n) + F(r^n) - F(r^{n-1}) + F(r^{n-1}) - F(r^{n-2}) + \cdots\cdots \\ &= (n+1)(N - r^n) + n(r^n - r^{n-1}) + (n-1)(r^{n-1} - r^{n-2}) + \cdots\cdots \\ &= (n+1)N - (r^n + r^{n-1} + r^{n-2} + \cdots\cdot) = n \cdot N + O(N). \end{aligned}$$

Lemma 2.—$K(N) = nN/r + O(N)$.

x will contain b in the tth place if and only if

$$s \cdot r^t + b \cdot r^{t-1} \leqslant x < s \cdot r^t + (b+1) \cdot r^{t-1}.$$

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[1] In their *Introduction to the Theory of Numbers*, Hardy and Wright give a slightly different definition which seems to contain superfluous conditions.

[2] *The Journal of London Math. Soc.*, 1933, **8**, 254–60.