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Physical nature of the event horizon

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Abstract. It is shown by arguments based on the uncertainty principle that large fluctuations of the metric occur near the event horizon of a Schwarzschild black hole on a scale much larger than Planck length. The width of the transition layer about the event horizon and the associated surface tension are also estimated.

Keywords. Event horizon; transition layer; surface tension.

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1. Introduction

In the accompanying article [1], hereafter referred to as I, it is shown how discontinuous metrics can arise as the limit of a sequence of relatively smooth metrics. It is shown that discontinuous metrics usually have an associated surface tension and/or surface energy which may, in some cases, become formally infinite. In the physically interesting case discussed in I, the surface tension goes as a quadratic in $\ln(l)$ where l is the size of the transition layer in which the exterior (Schwarzschild) metric goes over into the interior metric; this becomes infinite as $l \to 0$. It is argued that quantum effects provide a natural cutoff $l \sim L_{\rm Pl}$ where $L_{\rm Pl} = \sqrt{\hbar G}$ is the Planck length. (We shall be using units in which the velocity of light and the Boltzmann constant are unity.) We have argued elsewhere [2] (henceforth called II), that this is the case relevant to a black hole. In this paper, we shall use thermodynamic and uncertainty principle arguments to estimate the width of the transition layer; as a byproduct, we shall also obtain an explicit expression for the surface tension. This, then, will substantially complete a semi-classical description of the Schwarzschild black hole.

Our main results will be presented in § 2. We conclude in § 3 with some comments.

2. Width of the transition layer

The limiting discontinuous metric of a Schwarzschild black hole was guessed in II:

$$ds^2 = A dt^2 - B dR^2 - R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

where ($\Theta(x)$ is the usual step function)

$$A = -\Theta(R_g - R) + (1 - R_g/R)\Theta(R - R_g) \tag{2}$$

$$\frac{1}{B} = \left(1 - \frac{R^2}{R_g^2}\right)\Theta(R_g - R) + \left(1 - \frac{R_g}{R}\right)\Theta(R - R_g) \tag{3}$$

Here $R_g = 2GM$ is the gravitational (Schwarzschild) radius of the black hole whose mass is M. It is then possible to define smooth functions whose limits give (2) and (3). Using A and B to denote them also, we write

$$\frac{1}{B} = 1 - \frac{3}{R_g^2 R}\int_0^R x^2 f(x)\,\mathrm{d}x \tag{4}$$

$$A = \mathcal{E}(R - R_0)\exp\left(\left(\frac{1}{2} + F(R)\right)\ln\frac{1}{B}\right) \tag{5}$$

where $\mathcal{E}(x) = \Theta(x) - \Theta(-x)$ and $R_0 \to R_g$ as $l \to 0$. The two functions $f(x), F(x)$ are smooth functions with limits

$$f(R) \to \Theta(R_g - R), \tag{6}$$

$$F(R) \to \tfrac{1}{2}\mathcal{E}(R - R_g), \tag{7}$$

and certain other properties discussed in I. One example is,

$$f(R) = f_0(R) + \alpha f_0(R)(1 - f_0(R)), \tag{8}$$

$$F(R) = \frac{1}{2}\left[1 - 2f(R) - \frac{f(R)(1 - f(R))}{f(R_0)(1 - f(R_0))}(1 - 2f(R_0)\right], \tag{9}$$

$$f_0(R) = \frac{1}{1 + \exp(R - \overline{R}_g/l)}, \tag{10}$$

($\alpha, \overline{R}_g$ are fixed in I).

These equations introduce the parameter l for this choice of the functions f and F; the definition of l for more general functions is analogous. It is then shown in I that these metrics imply via Einstein equations a surface tension $\sigma = \sigma_c$ given by

$$\sigma_c = \frac{1}{8\pi G R_g}\left(E - C\ln\frac{R_g}{l} + D\left(\ln\frac{R_g}{l}\right)^2\right) \tag{11}$$

This surface tension, being a purely general relativistic effect, is entirely classical; we have put a subscript c to show this. The quantities E, C, D are pure structure dependent numbers; they depend on the form of the functions f and F in an essential way. This means that they are determined by microscopic physics which, in this context, means Planck level physics. In eq. (11) we necessarily have

$$D > 0 \tag{12}$$

in our model while C can have either sign. The classical nature of eq. (11) restricts l by the inequality $l \gg L_{\mathrm{Pl}}$; on the other hand $l \ll R_g$ for the surface tension to be meaningful.

At distances $l \lesssim L_{\mathrm{Pl}}$ (in particular as $l \to 0$) the above equation is necessarily inaccurate. Significant variation of the metric coefficient A in such a small distance must bring in purely quantum effects which produce a large surface tension σ_q. This surface tension must depend on $\hbar$ and l, and only these, if $l \ll L_{\rm Pl}$. Then we must have

$$\sigma_q = \frac{F\hbar}{l^3} \quad \text{as} \quad \frac{l}{L_{\rm Pl}} \to 0 \tag{13}$$

where F is a numerical constant. (In quantum mechanics, this type of constant varies from case to case.) Equation (13) is, in effect, a statement of the uncertainty principle in this case

$$\sigma l^3 \sim \hbar \tag{14}$$

(σ, here, should be read as the quantum uncertainty in surface tension). Equations (13) and (14) are essentially hypotheses concerning the elementary processes in quantum gravity. They imply that, at this level ($l \ll L_{\rm Pl}$), these processes are scale independent and so depend only on $\hbar$ and l but not on $L_{\rm Pl}$. Equation (13), then, follows from dimensional considerations. A word about the meaning of l is in order. For $l \ll L_{\rm Pl}$, l cannot be identified as the physical width of the transition layer. Its meaning is essentially mathematical. (We shall see, however, that the final equilibrium value of l is sensibly larger than $L_{\rm Pl}$). In principle, there could be terms with an additional factor $\ln(l/L_{\rm Pl})$, for example, from higher order effects; we assume that they are absent or unimportant.

We shall now combine the two expressions to obtain a candidate for the complete surface tension

$$\sigma = \sigma_q + \sigma_c \equiv \sigma(l). \tag{15}$$

This kind of interpolation formula is widely used in problems based on uncertainty principle [3]. It does not tell us what the (equilibrium value of) surface tension is because we do not know what value of l should be used in it.

We can obtain that if we use thermodynamic reasoning. Associated with the surface tension there is a surface free energy

$$\mathcal{F}_{\rm surf} = \sigma\mathcal{A}, \tag{16}$$

where $\mathcal{A}$ is the effective area of the event horizon of the black hole. The effective area of the event horizon is different from the usual (say $4\pi R_g^2$) area because of gravitational effects and is not known, but it will be seen that this does not matter as long as it is insensitive to changes in l. It is physically obvious that is so in the present case. Equation (16) together with eq. (15) gives us an expression for $\mathcal{F}_{\rm surf}$ off equilibrium. From (16) we see that there is a surface contribution to the thermodynamic potential $\Omega = \mathcal{F} - \mu N$:

$$\Omega_{\rm surf} = \sigma\mathcal{A}. \tag{17}$$

In equilibrium, the total potential Ω must be minimised. As far as variations in l are concerned, this implies that the surface tension σ is minimum in equilibrium, it being readily seen that all other contributions to Ω are relatively insensitive to l which is the relevant generalised co-ordinate here.

It will now be seen that $F > 0$; otherwise, there is no lower bound as $l \to 0$. Then, we have at equilibrium ($l = l_0$)

$$\left(\frac{\mathrm{d}\sigma(l)}{\mathrm{d}l}\right)_{l=l_0} = -\frac{3F\hbar}{l_0^4} + \frac{1}{8\pi G R_g}\left(\frac{C}{l_0} - \frac{2D}{l_0}\ln\frac{R_g}{l_0}\right) = 0. \tag{18}$$

For $l_0 \ll R_g$ this requires $C > 0$. We, then have the equation,

$$\frac{24\pi F L_{\rm Pl}^2 R_g}{l_0^3} = C - 2D \ln \frac{R_g}{l_0}. \tag{19}$$

We define a number $x_0(R_g)$ by

$$l_0 = \left(\frac{24\pi F}{C}\right)^{1/3} x_0(R_g) L_{\rm Pl}^{2/3} R_g^{1/3}, \tag{20}$$

to get

$$\frac{1}{x_0^3} = 1 + \frac{2D}{3C} \ln\left(\frac{24\pi F}{C}\right) - \frac{4D}{3C} \ln \frac{R_g}{L_{\rm Pl}} + \frac{2D}{C} \ln x_0. \tag{21}$$

If D/C is large and $(24\pi F/C)$ is, as expected, a number of order unity, a solution of this equation can occur only for large $x_0 \sim R_g^{2/3} L_{\rm Pl}^{-2/3}$; that being unacceptable we assume that D/C is small. Then a solution correct to the order D^2/C^2 can be quickly obtained:

$$x_0 \cong \left[1 + \frac{2D}{3C} \ln \left(\frac{24\pi F}{C}\right) - \frac{4D}{3C} \ln \frac{R_g}{L_{\rm Pl}} - \frac{2D}{3C} \ln\left(1 + \frac{2D}{3C} \ln\left(\frac{24\pi F}{C}\right) - \frac{4D}{3C} \ln \frac{R_g}{L_{\rm Pl}}\right)\right]^{-1/3} \tag{22}$$

It will be assumed that, for R_g of astrophysical interest, say $M \lesssim 10^{12}$ solar masses, this remains adequate. We see that x_0 is a rather slowly varying number of order unity. From now on, we shall put $D = 0$ to simplify expressions. Then $x_0 \equiv 1$. We then find the following expressions for the equilibrium surface tension which, when written in terms of the Hawking temperature $T = \hbar/(4\pi R_g)$, reads

$$\sigma_0 = \frac{T}{L_{\rm Pl}^2} \left(\alpha + \frac{C}{3} \ln \frac{T}{M_{\rm Pl}}\right), \tag{23}$$

where $M_{\rm Pl} = \sqrt{(\hbar/G)}$ and

$$\alpha = \frac{C}{6}\left(1 + \ln\left(\frac{6F(4\pi)^3}{C}\right)\right) + \frac{E}{2}. \tag{24}$$

Equations (20) and (23) constitute the central results of this paper. They show that the formalism gives sensible results; for example, $\sigma_0 \to 0$ as $T \to 0$.

3. Conclusions

This paper is in continuation of a programme to understand black hole thermodynamics. The main results were obtained in II but the discontinuity of the metric made it impossible to make unambiguous sense of Einstein equations. It was, therefore, felt necessary in I to invent a continuous metric which interpolated between the exterior and the interior metrics and which, in the limit as $l \to 0$, went over into the classical black hole metric. The associated surface tension was found to be logarithmically singular as $l \to 0$. Again, it was felt that quantum effects will provide, hopefully, a cutoff at some level of the order of the Planck length so that the associated surface tension could be meaningful. This is what has been achieved here. It is seen that if the transition comes about too abruptly, the surface tension, and hence the associated free energy, rises steeply by quantum kinematic effects. If the transition occurs too gradually, the surface tension again rises, but slowly, because of general relativistic dynamic effects. The actual transition occurs, on the average, at a trade-off point and determines the width of the transition layer and the equilibrium value of the surface tension. It is easy to see that if the general relativistic expression σ_c for the surface tension remains finite as $l \to 0$, the quantum corrections to it are, in effect, negligible; otherwise, as in the present case, quantum corrections are important and may even be dominant. The estimate $l_0 \sim L_{\rm Pl}^{2/3} R_g^{1/3}$ makes l_0 much larger than $L_{\rm Pl}$ because of our assumption $R_g \gg L_{\rm Pl}$ throughout. Though obviously possible, this was not anticipated. This result reminds one of Landau–Ginzberg theory [4]. It should be noted, furthermore, that the minimum is not a deep one, but has a width which is of the same order as l_0. This implies that there will be large quantum fluctuations in the metric on a scale $l_0 \gg L_{\rm Pl}$. The surface tension, therefore, is not really definite in a classical sense: quantum fluctuations are of the same order as the classical estimate. It is also seen, as noted in I, that all scales up to Planck scale are relevant for the details of this problem. In a qualitative sense we can say that the surface tension $\sigma \sim \hbar/l_0^3$ and is essentially quantum mechanical. This was vaguely suspected earlier in II, but the arguments were not precise. We see that the event horizon is a "critical point" of the metric. This will bedevil all future discussions of surface phenomena and should also be kept in mind when discussing II.

We see, then, that Einstein equations are not valid in the vicinity of the event horizon. One indication of this comes from (23). Associated with the surface tension there is a surface energy e given by the thermodynamic identity

$$e = \sigma_0 - T\frac{\mathrm{d}\sigma_0}{\mathrm{d}T}. \tag{23}$$

This e is the singular part of the energy-momentum tensor T_0^0 and is given by

$$e = -\frac{C}{3}\frac{T}{L_{\rm Pl}^2}. \tag{24}$$

Einstein equations, however, predict zero surface energy in the classical limit, despite the fact that the expression for e given in (24) is independent of $\hbar$. A finite e will lead to a discontinuity in $1/B$ which, in turn, will lead to the occurrence of more complicated singularities (such as derivatives of δ-function or worse) in the energy-momentum tensor, contrary to the assumption made in I. On the other hand, eq. (23) may also fail for the same reason.

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