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On a Nonlinear and Lorentz-Invariant Verson of Newtonian Gravitation

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Abstract. The Newtonian theory of gravitation is modified to include the gravitational energy as a source of gravitational potential, thus making the theory self-coupled and nonlinear. The modified theory can be derived from a Lorentz-invariant action principle. The Kepler problem is discussed in this theory and it is shown that the perihelion of the orbit steadily precesses. The rate of precession is, however, insufficient to account for the observed precession of the perihelion of Mercury. The differences from the Newtonian theory for the bending of light and the gravitational redshift of spectral lines are shown to be marginal.

Key words: Newtonian gravitation—Lorentz-invariant generalization

1. Introduction

In the language of field theory the Newtonian law of gravitation is described by Poissons' equation

$$\nabla^2\phi = -4\pi G\rho \tag{1.1}$$

where ϕ is the potential and ρ the matter density. The gravitational force per unit mass is given by

$$\mathbf{F} = \nabla\phi \tag{1.2}$$

If we wish to reconcile Newtonian gravitation with special relativity we have to make Equation (1.1) Lorentz invariant. Thus ∇^2 on the left-hand side is replaced by the wave operator

$$\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{1.3}$$

What should we replace the right-hand side by? There are two choices open to us, both dictated by the equivalence of mass and energy in special relativity.

The first choice is to recognize ρc^2 as the time-time component of the energy momentum tensor T_{ik}. In this case, we cannot restrict the theory to a scalar form. The function ϕ must be replaced by a tensor ϕ_{ik} whose time-time component satisfies

$$\frac{1}{c^2}\frac{\partial^2\phi_{00}}{\partial t^2} - \nabla^2 \phi_{00} = \frac{4\pi G}{c^2} T_{00}. \tag{1.4}$$

We feel that a modification along these lines would change the Newtonian character of the theory in the sense that a simple scalar potential has been replaced by a second-rank tensor.

The second alternative preserves ϕ as a scalar and replaces ρ by the scalar T obtained by taking the trace of T_{ik}. Thus Equation (1.1) is changed to

$$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \nabla^2 \phi = \frac{4\pi GT}{c^2}. \tag{1.5}$$

We will start with Equation (1.5) as the first step in our modification of Newtonian theory.

The next step will bring the theory closer in spirit to general relativity. In this step we introduce the notion that T includes all sources of energy. Thus T_{ik} may include electromagnetic energy tensor and any other energy tensor that happens to be relevant. This raises the question: 'What about gravitational energy?'

A little consideration shows that the inclusion of gravitational energy on the right-hand side of Equation (1.5) will make the theory nonlinear. Naive Newtonian considerations show that the gravitational energy density is given by

$$\rho_g c^2 = -\frac{1}{8\pi G} | \nabla \phi |^2. \tag{1.6}$$

Thus, inclusion of (1.6) on the right-hand side makes the potential equation nonlinear.

Apart from conceptual considerations, another reason for investigating such a theory is to see if the resulting equations give a better agreement with the solar system tests than the original Newtonian theory. Before considering these applications, it is desirable to place the heuristic considerations above on a more formal footing. We therefore approach the problem with an action principle.

2. The action principle

Consider first the action principle which gives rise to Equation (1.5). We will use the Minkowski line element

$$ds^2 = \eta_{ik} dx^i\, dx^k \tag{2.1}$$

with η_{ik} = diag (+1, − 1, −1, −1). The coordinates x^1, x^2, x^3 are the Cartesian space coordinates while, $x^0 = ct$ is the time coordinate. Denote $\partial\phi/\partial x^i$ by ϕ_i, and use the summation convention. Then the action is given by

$$J = \frac{1}{16\pi Gc}\int \phi_l \phi^l \, d^4x + \int \frac{T\phi}{c^3}\, d^4x. \tag{2.2}$$

The energy tensor for ϕ is given by

$$T_{ik}(\phi) = -\frac{1}{8\pi G}\left[\phi_i \phi_k - \tfrac{1}{2}\eta_{ik}\phi^l \phi_l\right], \tag{2.3}$$

so that

$$T(\phi) = \frac{1}{8\pi G}\phi^l \phi_l. \tag{2.4}$$

If we now imagine T in the second term to include $T(\phi)$, we have to write

$$T = T_m + T(\phi), \tag{2.5}$$

T_m being the contribution from matter alone.

If we do not take note of ϕ dependence in Equation (2.5) and perform a variation of ϕ, $\delta J = 0$ gives us

$$\Box\phi = \frac{4\pi GT}{c^2}, \tag{2.6}$$

which is the same as Equation (1.5). However, this procedure is obviously incorrect. We should vary ϕ in $T(\phi)$ also, thus writing

$$J = \frac{1}{8\pi Gc}\int \left(1 + \frac{\phi}{c^2}\right)\phi_l \phi^l \, d^4x + \frac{1}{c^3}\int T_{\mathrm{m}}\phi \, d^4x, \tag{2.7}$$

which gives, from $\delta J/\delta\phi = 0$, the following source equation for ϕ:

$$\left(1 + \frac{\phi}{c^2}\right)\Box\phi - \frac{1}{2c^2}\phi_l \phi^l = 4\pi G T_{\mathrm{m}}/c^2. \tag{2.8}$$

These equations are different from those derived by Nordstrom (1913) from other considerations.

In static case, this reduces to the modified Poisson equation

$$\left(1 + \frac{\phi}{c^2}\right)\nabla^2\phi - \frac{1}{2c^2}(\nabla\phi)^2 = -4\pi G\rho_{\mathrm{m}}. \tag{2.9}$$

We will now consider the solution of Equation (2.9) outside a spherical mass M.

3. Gravitational field outside a spherical mass

Since outside a spherical mass M, $\rho_m = 0$, Equation (2.9), with the assumption of spherical symmetry and with ϕ as a function of the radial coordinate r only, becomes

$$\left(1+\frac{\phi}{c^2}\right)\left[\frac{d^2\phi}{dr^2}+\frac{2}{r}\frac{d\phi}{dr}\right]-\frac{1}{2c^2}\left[\frac{d\phi}{dr}\right]^2=0. \tag{3.1}$$

This non-linear equation can be solved with the substitution $\mu = \text{In}\left(1+\frac{\phi}{c^2}\right)$. A simple calculation gives

$$e^{\mu/2}=\left(A+\frac{B}{r}\right), \tag{3.2}$$

A, B beingc onstants. Assuming that $\phi \sim GM/r$ as $r \to \infty$, we get $A = 1$ and $B = \text{GM}/2c^2$ so that

$$\phi=\frac{GM}{r}+\frac{G^2M^2}{4c^2r^2}. \tag{3.3}$$

Thus, in addition to the inverse square force, we also have an inverse cube force which varies as $G^2M^2\ 2r^3/c^2$.

With this force we can work out the precession of the perihelion of a planetary orbit. In polar coordinates the equations of motion are

$$\dot{r}-r\dot{\theta}^2=-\frac{GM}{r^2}-\frac{G^2M^2}{2c^2r^3}, \tag{3.4}$$

$$\dot{r}^2\dot{\theta}=\text{h (= constant)}. \tag{3.5}$$

To solve these, substitute $u = 1/r$ and $R_0 = 2GM/c^2$. Then Equations (3.4) and (3.5) give

$$\frac{d^2u}{d\theta^2}+\left[1-\frac{R_0^2c^2}{8h^2}\right]u=\frac{R_0c^2}{2h^2}. \tag{3.6}$$

Define a new angular coordinate $\bar{\theta}$ by

$$\bar{\theta}^2=\left[1-\frac{R_0^2c^2}{8h^2}\right]\theta^2. \tag{3.7}$$

When the orbit is completed once from perihelion to perihelion, $\bar{\theta}$ increases by 2π. But θ increases by

$$\frac{2\pi}{\left[1-\frac{R_0^2c^2}{8h^2}\right]^{1/2}}\simeq 2\pi+\frac{\pi R_0^2c^2}{8h^2}. \tag{3.8}$$

For an orbit of semi-latus rectum l and eccentricity e, we have in the Newtonian case

$$h^2 = GMl \tag{3.9}$$

and the orbital period

$$T = \frac{2\pi l^2}{(1-e^2)^{3/2}\, h}. \tag{3.10}$$

From Equation (3.8) we get the rate of perihelion precession as

$$\tilde{\varpi} = \frac{\pi R_0^2 c^2}{8h^2 T} = \frac{\pi GM}{2lTc^2}. \tag{3.11}$$

This is 1/12 of the general-relativistic value

$$\tilde{\varpi}_{\mathrm{GR}} = \frac{6\pi\, GM}{lTc^2}. \tag{3.12}$$

In the above analysis, we did not include the effect of special-relativistic terms. It is well known (see Stephenson and Kilmister 1958) that the special-relativistic terms give–in the usual linear Newtonian theory–a precession rate

$$\tilde{\varpi}_{\mathrm{SR}} = \tfrac{1}{6}\, \tilde{\varpi}_{\mathrm{GR}}. \tag{3.13}$$

A similar analysis carried out in our non-linear theory gives the total effect as the sum of (3.11) and (3.13) in the case of Mercury. That is, the total effect is $\tilde{\varpi}_{\mathrm{GR}}/4$. The calculation of bending of light in the present theory yields a bending angle

$$\theta = \frac{2GM}{Rc^2}\left[1 + \frac{\pi GM}{8Rc^2}\right], \tag{3.14}$$

where R is the radius of the gravitating object. Although the bending is marginally greater than the Newtonian angle, it falls far short of the actual bending observed for microwaves grazing the solar surface. The gravitational redshift formula also gives a correction of the order GM/Rc^2 to the Newtonian value.

4. Conclusion

Our calculations show that the logical modification of Newtonian gravitation as a Lorentz-invariant self-coupled scalar field theory fails to satisfy the solar system tests, although it gives a better agreement with observations than the original Newtonian theory.

References

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