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Chandrasekhar, Black Holes, and Singularities

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1. Historical background

The dilemma that was presented to the scientific world by Chandrasekhar's early work (1931) on the existence of a maximum mass for white dwarf stars took some while to be fully appreciated. There were some, such as Eddington, who did seem to understand the alarming implications of Chandra's conclusions. Assuming the correctness of the relativistic equations of state, it seemed that a white dwarf star of mass more than about 1.4 of a solar mass would have to collapse inwards, its density increasing indefinitely as the body approached a singular configuration at the centre. However, Eddington himself regarded this as a *reductio ad absurdum*, concluding, instead that there must be something wrong with Chandra's use of the relativistic equations. Eddington supposedly had in mind that some new physical principles must come into play in order to save the star, perhaps such as embodied in his own approach to a deeper fundamental theory (Eddington 1946). Taken at the level of phenomena at which Chandra's discussion was intended to apply, there is no doubt that Chandra's analysis was the correct one, as the experts seemed to have appreciated even at that time (at least privately), despite the weight that Eddington's authority attached to the contrary view. Yet Eddington had a point: the impossibility of an equilibrium state would lead to the star's unstoppable collapse. Would this collapse continue until the star becomes so compressed that it reaches its Schwarzschild radius ($r = 2m$), thought at that time to be a dimension at which the very metric structure of space-time becomes singular? In any case, as the star continues to collapse radially inwards, it appears that it should reach a state where the density becomes infinite. And, according to Einstein's general theory of relativity, infinite matter density would in itself imply a singularity in space-time structure. Thus, it would seem that even if one accepted the procedures of conventional physics completely, with the conclusions that Chandra so strongly argued for, the unending collapse of the star would lead to a situation in which those very procedures would ultimately have to be abandoned.

By nature conservative in his approach, Chandra would certainly not have been attracted to a radical kind of speculation that had so occupied Eddington in his later years. In the 1930s, even general relativity was not regarded as a worthy activity for aspiring astrophysicists to devote themselves to. In any event, for whatever reason, Chandra chose not to mount a direct attack on the problem of gravitational collapse.

In that regard, he bided his time. Indeed, there were many other issues that would occupy his attentions for a quarter of a century!

In around 1960, having completed his works on stellar structure and stellar dynamics, on radiative transfer, and on the equilibrium or stability of various structures of astrophysical interest, he finally resolved to enter into a study of general relativity. He attended the 1962 International Conference on General Relativity and Gravitation in Warsaw as a 'student', in order to attain an overview of current research in the subject. Even after he had thoroughly prepared himself, he did not directly address the issue of the fate of a collapsing star. He was more concerned, first, with the effects of general relativity on the stability of gravitating bodies, and then with general-relativistic corrections to the Newtonian dynamics of collections of masses, including the effects of gravitational radiation damping. These studies were relevant to the issue of how considerations of general relativity might affect the onset of gravitational collapse, not with the result of the collapse itself. In short, the overall conclusion was that the influence of general relativity accelerates gravitational collapse somewhat, over and above the various influences of other physical effects.

So was collapse right down to a singularity of some sort the inevitable conclusion? For a white dwarf star, there might still be a number of other possibilities open to it. It might, for example, have the opportunity to find ultimate rest as a neutron star. But it was already an implication of Chandra's analysis (cf. Landau's simplified approach of 1932) that it applied also to the relativistic equations of state for neutron matter. The analysis was carried out in detail by Oppenheimer and Volkoff (1939), the conclusion at that time being that the maximum mass for a neutron star was a little smaller, even, than for a white dwarf. Allowing for the fact that the neutrons themselves can become converted to other massive particles (not known at the time), somewhat larger limiting masses were subsequently suggested. On the other hand, very general considerations, based on fundamental principles such as causality, allowed the conclusion to be made that there was an absolute overall limit of not much more than 3 solar masses. There appears to be no clear agreement as to what the actual limit is, but observations seem to suggest something like about 1.4 solar masses. Accordingly, no final solution to the problem of gravitational collapse is to be found along these lines.

Already in 1939, Oppenheimer and Snyder had faced this problem squarely, considering the situation of a collapsing spherically symmetrical dust cloud, of uniform density, treated according to Einstein's general relativity. In their description, they provided the first explicit model of a collapse to what is now called a 'black hole'. They showed that although the so-called 'Schwarzschild singularity', which occurred at radius $r = 2m$, is not actually a singularity –what is now called the 'horizon' – there is still a space-time singularity at the centre, where the density of the dust indeed becomes infinite. Ironically, it was Eddington himself who first published, in 1924, a metric form for the spherically symmetric Schwarzschild space-time in which the 'Schwarzschild singularity' is exhibited in its true nature as a null hypersurface (actually two null hypersurfaces). However, Eddington appeared not to appreciate what he had himself done. Before Oppenheimer and Snyder, Lemaîitre (1933) had appreciated that the 'Schwarzschild singularity' could be locally eliminated by a coordinate change but he had not provided an overall picture of collapse to a black hole. Later Synge (1950) and others found the complete extension of the Schwarzschild solution.

Since there is still a singularity in the Oppenheimer-Snyder collapse model (at $r = 0$), the Chandrasekhar dilemma is not removed by their collapse picture. However many people remained unconvinced that this description would necessarily be the inevitable result of the collapse of a star too massive to be sustainable as either a white dwarf or neutron star. There were a number of good reasons for some scepticism. In the first place, the equations of state inside the matter were assumed to be those appropriate for pressureless dust, which is certainly far from realistic for the late stages of stellar collapse. Moreover, the density was assumed to be constant throughout the body. With realistic material, there are many alternative evolutions to that described by Oppenheimer and Snyder. For example, nuclear reactions set off at the centre could lead to an explosion – a *supernova* - which might perhaps drive off sufficient mass from the star that a stable equilibrium configuration becomes possible.

Most serious of all was the assumption of exact spherical symmetry. Since, in this picture, all the material of the body is aimed directly at the central point, a resulting density singularity could easily be the result merely of this fortuitous focussing. It could be expected that the introduction of even the slightest perturbation away from spherical symmetry might cause most of the inward falling particles of the body's material to miss the central point, so that even though the density might get very large there, it might well not diverge to infinity.

Indeed, as late as 1963, the Russian school of Lifshitz and Khalatnikov (1963) had claimed that the 'generic' solution of the Einstein equations ought to be free of singularities. However, it became evident that there must be a significant error in their work when the early singularity theorems were established (Penrose 1965; Hawking 1966; Hawking & Penrose 1970). In fact, such an error was found and corrected by Belinskii (cf. Belinskii, Khalatnikov and Lifshitz 1970, 1972), their final conclusion being that singularities could occur in generic solutions after all. (Lifshitz and Khalatnikov had omitted a certain degree of freedom in their expansions.)

In any case, it is difficult to form any firm conclusions about the final stages of gravitational collapse from the kind of power series analysis that the Russian school had adopted. Perturbation analysis is not well suited to the extreme situations that would be expected to arise when a space-time approaches a singular state. Moreover, the possibility of finding explicit solutions of sufficient generality to describe what happens in a realistic collapse could be virtually ruled out. For reasons such as these, the approach adopted in the singularity theorems was completely different. Instead of attempting to work out what happens in detail to a collapsing star, general overall considerations, of a largely topological nature were employed so as to derive general properties of the solution. In essence, the conclusions were all of a negative character, in the sense that they ruled out certain things, showing that they cannot happen – rather than showing what actually does happen.

What cannot happen, according to these theorems, is an ordinary singularity-free evolution – assuming that equations of state satisfy a reasonable energy – positivity condition – if the collapse reaches a certain point of no return. This 'point of no return' can be characterized as the existence, in the space-time, of what is called a 'trapped surface'. This is a compact spacelike 2-surface whose null normals converge into the future (which means that if a flash of light were to be emitted at the surface, then the area of any element of cross-section of the emitted light rays must decrease; cf. Penrose 1965). Another slightly different way of specifying an appropriate 'point of no return' is the existence of a point in the space-time through which all light rays into the future begin to reconverge somewhere (cf. Hawking & Penrose 1970). By an 'ordinary singularity-free evolution', I mean that it must be possible to continue the space-time (non-singularly) into one in which all null and/or timelike geodesics have infinite affine length and for which appropriate causality conditions are maintained (such as the existence of a global Cauchy hypersurface or merely the absence of closed timelike curves together with a condition that the space-time is in some mild sense 'generic'). There are slight differences in the details of these various conditions, depending upon which singularity theorem is being appealed to. (For further information, see Hawking & Penrose 1970; Hawking & Ellis 1973).

The upshot of all this is that once a trapped surface (or reconverging light cone) has formed, there is no way, within the scope of existing physical laws, to extend the space-time indefinitely. This is the ultimate dilemma that Eddington was, in effect, shying away from. But why should we expect a trapped surface to arise in any case? In the Oppenheimer-Snyder picture, just after the collapsing matter shrinks within the Schwarzschild radius at $r = 2m$ (the event horizon), there are trapped surfaces in the space-time region immediately surrounding the body. Since the trapped-surface condition is, by its very nature, something undisturbed by (adequately small) finite perturbations, the essential issue is whether or not a collapsing body, or collection of bodies is ever likely to reach the vicinity of its Schwarzschild radius – and just beyond. Chandra's work on the influences of general relativity on the stability of massive bodies, let alone his much earlier initiation of the entire line of thinking concerning the maximum mass of bodies held apart by relativistic degeneracy pressure, provided strong support for the view that unstoppable collapse to the neighbourhood of the Schwarzschild radius was probable.

Nevertheless, these arguments are not totally convincing, owing to the lack of complete information about details of the internal state of a star, where the density might exceed, by several orders of magnitude, even that inside an atomic nucleus. However, if it is matters of principle that we are concerned with, these details are not important. One can envisage situations in which trapped surfaces will arise even for densities that are as low as, say, that of air. By making the total mass of the collapsing system large enough, the density at which the body crosses its

Schwarzschild radius can be made as small as we please. At the centres of large galaxies there would be collections of ordinary stars which, if they were to find themselves in a small enough region all at once, would be surrounded by a trapped surface even though the stars are not yet in contact, so the total density is less than stellar density. This follows from very basic considerations of general relativity – even directly observed ones – concerning the focussing power of mass density on light rays. Rather easier than the trapped- surface condition is to use the condition of a reconverging light cone. Suppose that the radius of the region is 10^4 km. The light rays emerging from a point at the centre of the star cluster would encounter sufficient stellar material that they indeed begin to reconverge, as follows from a simple order-of-magnitude calculation. (See Penrose 1969, p.266, for a simple description of this idea.)

It follows from this, and the singularity theorems, that if conventional physical ideas hold true, we are forced into having to face up to the occurrence of space-time singularities. We must ask what is the nature of these singularities and what are we to do about them. In fact, the singularity theorems are almost completely silent about the nature of the singularities themselves. The theorems are simply existence theorems, and say almost nothing about the location of the singular regions, let alone anything about their detailed nature. From later work (e.g., Clarke 1993, and also the earlier work of Belinskii, Khalatnikov and Lifshitz 1970, 1972) there were certain very incomplete indications as to the singular behaviour of the Weyl curvature tensor. Enough is known to suggest that the structure of the singularities in a generic collapse will be quite different from that which is encountered in the big bang (divergent Weyl curvature in the former, essentially vanishing Weyl curvature in the latter – a feature intimately connected with the second law of thermodynamics, cf. Penrose 1979). But very little is known in detail. As to 'what we are to do about them', it is clear that these singularities take us outside the domain of classical general relativity. Without the appropriate union between general relativity and quantum mechanics being to hand, we are presented with an impasse.

The existence of singularities does not, however, imply the existence of black holes. This deduction requires the additional assumption of what is called 'cosmic censorship'. Cosmic censorship (Penrose 1969, 1978) asserts that naked singularities will not occur in a generic gravitational collapse. Such a singularity, roughly speaking, would be one that could be seen by an outside observer. Cosmic censorship would imply that the region lying to the future of all singularities resulting from gravitational collapse cannot reach future null infinity *I*+. The boundary of the entire region which cannot be connected to *I*+ by a causal (i.e. timelike or null) curve defines the (absolute) *event horizon H.* We thus see that a gravitational collapse resulting in singularities subject to cosmic censorship will lead to the existence of a *horizon H* which 'shields' all the singularities from directly revealing themselves from the outside world. This horizon describes a black hole. (See Penrose 1973)

Various theorems (Israel 1967; Carter 1970; Robinson 1975; Hawking 1972) can now be called into play, where it is assumed that the Einstein vacuum (or Einstein-Maxwell) equations hold from the neighbourhood of *H* out to infinity, after the collapsing body has fallen through. The upshot of these theorems is that if the space-time in this region settles down to a stationary configuration, then it is described by the Kerr metric (Kerr 1963), completely defined by two parameters *m* and *a* (or, in the Einstein-Maxwell case, the Kerr-Newman metric (Newman *et al.* 1965), defined by three parameters *m*, *a*, and *e*).

2. Chandra's work on perturbations of black holes

It is a remarkable fact that black holes, when they settle down to an exactly stationary configuration have such a precise and explicit mathematical description. Chandra never ceased to be impressed by this fact, remarking at the beginning of his Prologue to *The Mathematical Theory of Black Holes* (1983):

> "The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well."

Chandra's study of black holes was based upon these stationary cases: the Kerr (and Kerr-Newman) space-times – together with the Schwarzschild (and Reissner-Nordström) specializations. Since these stationary metrics were known explicitly, he was able to study, in comprehensive detail, the first-order gravitational perturbations away from the stationary Kerr configuration – and gravitational-electromagnetic ones in the Reissner-Nordström case. Moreover, he provided a comprehensive treatment of the Maxwell and Dirac equation on a Kerr background. In all these cases, he found remarkable algebraic/differential relations which enabled him to separate and decouple the equations.

Chandra was a relative latecomer to the study of black holes. In his early work on white dwarf stars and the inevitability of their collapse when too massive, leading to his early realization of the dilemma referred to in section 1, Chandra had been far ahead of his time. But his assault upon the very problem that his early researches had thrown up was delayed until after most of the groundwork had been carried through by others. There were, of course excellent reasons for this. Chandra's monumental work on other topics had to be completed first. It was his way of working that he would devote himself in a single-minded way to one topic at a time, thereby achieving· the phenomenal thoroughness and depth that he strove for in each topic.

However, he also had a long-standing interest in general relativity and, specifically, in what came to be known as black holes. But until about 1974 (about the time when most experts in the subject had moved away from the classical theory and were turning to the implications of Hawking's discovery of black-hole radiation), he had not felt ready to embark upon his detailed assault upon the area of black holes.

In particular, he felt that he needed to complete what had been his most recent work – on the stability of rotating stars – before doing so. As it happens, in this work, he was studying perturbative general-relativistic effects, including the effects of gravitational radiation. Thus, this research provided him with a natural route into the study of perturbations of black holes. Indeed, the various researches that Chandra has undertaken should not be thought of as being totally independent of one another. In particular, in his rotating star stability work, as in so many other things that he achieved, he exploited his extraordinary ability with equations, and he already had much of the basic framework of ideas to hand, ready for his concerted attack on the problems of black-hole perturbation. Indeed, it appears to have been Teukolsky's separation of the equations for gravitational perturbations of a Kerr black hole (Teukolsky 1973) that stimulated Chandra's actual entry into the subject.

The physical motivations for the study of black hole perturbations came from a desire to know how such an object, initially in a stationary configuration, would react if slightly disturbed. Its response could involve the emission of gravitational waves, with another part of the disturbance disappearing into the hole. It would be argued that this response would be governed primarily by the structure of the black hole itself. The linear perturbations of the hole would be described by linear equations, and could therefore be analysed in terms of the appropriate 'modes'. These are not quite like the modes of vibration of a perfectly elastic body, because of the effective dissipation that occurs both because of gravitational radiation and through. loss into the hole itself. The frequencies are therefore complex, with imaginary parts describing the decay of the modes.

Chandra's first paper on black-hole perturbations (Chandrasekhar 1975) was concerned with the Schwarzschild black hole. At that time there already existed a thorough analysis of these perturbations, dating back to the work of Regge and Wheeler in 1957. In particular, Zerilli (1970) had shown that, splitting the perturbation into different components for the individual spherical harmonics, each component satisfies a Schrödinger equation, corresponding to that for a wave of time dependence $e^{i\sigma t}$ (with complex frequency σ having an imaginary part describe-ing the damping rate) and spatial dependence $Z(x)$, impinging on a potential barrier defined by a certain smooth potential function $V(x)$:

$$\left(\frac{d^2}{dx^2}+\sigma^2\right) Z = VZ.$$

The variable x is related to the standard Schwarzschild radial coordinate r by $x = r + 2m \log(r - 2m)$. The potential function V is a particular explicit function of r , depending on the mass m and the choice of spherical harmonic. (Units are always chosen so that the speed of light c and Newton's gravitational constant G are both unity.) Accordingly, the general perturbation problem for the Schwarzschild black hole can be reduced to that of finding the transmission and reflection coefficients of a simple one-dimensional barrier-penetration problem in quantum mechanics. On the other hand, Bardeen and Press (1973) had obtained a different set of equations to describe the same perturbations, and various relations between all these expressions and procedures had remained somewhat mysterious. Rederiving all these expressions in his own way, Chandra was able to understand and explain several of these mysterious features, most notably a curious relationship between the coefficients that arise from the odd- and even-parity (or, as Chandra preferred, axial and polar) perturbations.

Later, he sought to analyse the deeper reasons underlying such relationships, noticing that one could understand these in terms of the procedures of inverse scattering, according to which consistency conditions of the nature of the Korteweg-de Vries equation arise (Chandrasekhar 1982). In this way, Chandra provided an opening into the intriguing and mathematically fruitful area of integrable systems, and it is likely that the last word on these matters has by no means been heard (see comments in section 4).

The initial disturbance which causes a perturbation to the geometry of a stationary black hole could be caused by some physical object falling into the hole, or it could be taken to have the form of incoming gravitational (or gravitational-electromagnetic) waves, impinging on the hole from outside. Particularly in his later work on the subject, Chandra preferred to emphasize the latter viewpoint, this having the advantage that no foreign ingredients are imported into the system of equations under consideration, everything being described in terms of the Einstein vacuum equations (or Einstein-Maxwell equations). Of course, in an actual astrophysical situation, there would normally be some other source for a significant black-hole perturbation, but from the point of view of making the mathematical treatment self-contained, this approach has considerable advantages.

In a situation of this nature, there is an incoming component (from *I*–) and two 'outgoing' components, one which escapes out to infinity (*I*+) and another one which falls into the hole. As mentioned above, the situation is closely analogous to that which arises with one-dimensional potential scattering and barrier penetration in ordinary quantum mechanics – and for each separate spherical harmonic, the mathematical description is precisely of this form. There is an incoming wave train and an outgoing reflected wave train, accompanied by the part which passes through the potential barrier. In particular, there will be that situation which arises when the incoming influence is 'switched off' and the black hole 'rings' according to its natural frequencies. These modes are what are called 'quasi-normal modes', characterized by the fact that there is no component coming in from *I*– (and no component coming out from the interior of the hole – a geometrical impossibility in any case unless one allows for a delayed burst of radiation from the collapsing matter which originally produced the hole, which would have to have hovered, exponentially decaying, at the hole's horizon since its formation). Thus, each quasi-normal mode is composed only of a wave train escaping to *I*+ and a wave train falling into the hole. Each of these would be exponentially decaying modes (but badly behaved at spatial infinity owing to their exponential blow-up at infinite negative times). Chandra studied these modes for the Schwarzschild black hole in his 1975a paper with Detweiler (and, later, for the Kerr black hole), but he warned against too much reliance in these giving a complete characterization of the decay behaviour of black-hole perturbations and on too much faith being placed on the analogy with normal modes of an elastic material body. As far as I am aware, there are still unanswered questions concerning completeness and other issues for quasi-normal modes for black holes.

After his comprehensive treatment of the spherically symmetrical (i.e., Schwarzschild) case, Chandra moved on to his study of perturbations of the Kerr space-times. These possess axial symmetry and include angular momentum. It was, after all, the remarkable simplicity of general relativity's implication that stationary vacuum black holes have to be Kerr (or Schwarzschild) metrics that so attracted him to this area of research. The fact that Teukolsky (1973) was able to separate the radial parameter and the angular coordinate θ to obtain a pair of decoupled equations for the gravitational perturbations of the Kerr metric was a surprising additional bonus. Chandra rederived the Kerr metric in his own way (Chandrasekhar 1978a). With Detweiler (Chandrasekhar & Detweiler 1975b), he showed that Teukolsky's equations for the gravitational perturbations can be reduced to a (complex) one-dimensional wave equation of the type considered in the displayed equation above, with four possible potentials (but now depending on the frequency σ); moreover, they showed that the reflection and transmission coefficients are the same in each of the four cases, thereby illuminating some puzzling relationships. This work was continued in an important series of papers (Chandrasekhar 1976a,b, 1978a,b,c, 1979a,b, 1980, 1983) in which Chandra completed a very thorough and detailed analysis of the gravitational perturbations of a Kerr black hole. These papers are full of remarkable algebraic and differential relationships and identities, following on from those that had been established by Teukolsky and Starobinski.

Teukolsky (1973) had also separated the equations for scalar waves and electromagnetic fields on a Kerr background. Chandra treated both of these fields in two papers in 1976, again reducing the equations to the same type of one-dimensional wave-equation as above. He then showed (Chandrasekhar 1976c), ingeniously performing separation prior to decoupling, that the Dirac equation for the electron could also be separated and decoupled in a Kerr background. Teukolsky's work had already covered the case of a neutrino field

$$\nabla^{AB'}\nu_{B'} = 0,$$

but for a Dirac field with non-zero mass $2^{1/2}\ \mu$ in 2-component spinor form, we have the coupled pair

$$\nabla_{AA'}P^A + i\mu\ \bar{Q}_{A'} = 0$$
$$\nabla_{AA'}Q^A + i\mu\ \bar{P}_{A'} = 0$$

and the separation of these had proved to be a stumbling block. Chandra took special delight in the fact, as noted in half of a sentence at the end of his paper, that by taking the limit in which the black hole's mass is set to zero, one obtains the separation of Dirac's equation in oblate spherical coordinates in flat space-time – a feat which had not been achieved before.

In his study of the Dirac equation, Chandra showed that he had mastered the 2-spinor formalism, which is not particularly familiar to physicists generally, there being an almost universal tendency for them to phrase their discussions of the Dirac equation in terms of the (superficially simpler but ultimately more complicated) 4-spinor formalism. In this, and also in much of his work in gravitational and electromagnetic perturbations (and in his later work on colliding plane waves), Chandra exhibited a great facility with what has become known as the 'NP-formalism' (the method of spin coefficients), which with Ted Newman and I had developed in 1962 to handle general relativity - effectively by combining the 2-spinor calculus with a Ricci-rotation-coefficient type of formalism (Newman & Penrose 1962). Chandra had specifically invited me to give a series of lectures in Chicago, in the early 1970s, primarily on the subject of this formalism, but I had in no way anticipated the powerful use that he would ultimately make of it.

To this, I might add a personal note of some irony. Some time later, in response to a question from me as to why he had not gone further and adopted the somewhat more streamlined later GHP formalism (Geroch, Held and Penrose 1973) Chandra had remarked that this would not simplify his equations in the way that I had imagined, because of an awkward problem about normalizing the spinor dyads against one another. Consequently, when writing the book *Spinors and Space-Time* with Wolfgang Rindler, I went to some trouble developing a generalized version of the GHP formalism in which the normalization condition was removed. This resulted in some extra complications which I know caused certain of my colleagues some irritation. However, I was not deterred by this, partly because the original GHP formalism can be extracted from this without difficulty, but more particularly because I believed that this was all in an excellent cause because Chandra could then directly incorporate this extended formalism into simplifying his equations! Apparently Chandra did not agree that this was any help, so we are left with a formalism still (as far as I know) looking for a good application. (Actually, I am still not convinced that it cannot be used with effect in the kind of thing that Chandra was doing. Perhaps some brave soul will have a look at it sometime.)

Chandra's work in relation to the Kerr metric applied to a background space-time in which the Einstein vacuum equations hold. Returning to the case of spherical symmetry, but where now the presence of an electromagnetic field is allowed for, Chandra studied gravitational-electromagnetic perturbations away from a stationary Reissner-Nordström black hole (Chandrasekhar 1979b; Chandrasekhar & Xanthopoulos 1979). Again, separation and decoupling of the perturbations occurs, with apparently rather little - but sometimes subtle – change required from the vacuum Schwarzschild case.

There is, however, one important change which does take place when one passes from the Schwarzschild to the Reissner-Nordström black hole, a change which occurs also when one passes to the Kerr black hole, this being the acquisition of a Cauchy horizon. A space-traveller who falls into the hole would, upon crossing the inner (Cauchy) horizon, enter a 'new universe' were it not for the fact that a shell of infinitely intense radiation coming from outside the hole would be expected to be encountered there. Some early work (cf. Simpson and Penrose 1973; McNamara 1978a,b) had provided a strong indication of this, but a more complete analysis of the perturbation theory, provided by Chandra with James Hartle in 1982, convincingly demonstrated the inevitability of this phenomenon (Chandrasekhar & Hartle 1982).

The combination of angular momentum with electric charge in a black hole gives rise to the solution of the Einstein-Maxwell equations known as the Kerr-Newman Metric (Newman *et al.* 1965). Chandra also studied Kerr-Newman black holes and their perturbations, but he was disappointed that he was unable to separate the equations for the gravitational perturbations, and he finally set the problem aside. If Chandra was not able to do it, that in itself would seem to be reason enough to believe that separation is not actually possible. However, the analogies with the Kerr and Reissner-Nordström black holes are very strong, so it is hard to resist the temptation that some separation, perhaps of a much more complicated nature, might lie somewhere behind the scenes. (In connection with this, following Chandra's separation of the Dirac equation on a Kerr background, Page was able, in 1976, to extend this result to the Kerr-Newman background.) I shall return briefly to the question of separation in section 4.

As was his general method of working, Chandra virtually set the seal on nearly ten years of research into this topic – black holes and their perturbations – by writing a superb book on the subject: *The Mathematical Theory of Black Holes*, published by the Clarendon Press, Oxford in 1983. Unlike the situation with his earlier books, however, he did not leave black holes entirely alone after that (cf. Chandrasekhar 1984, 1989; Chandrasekhar & Xanthopoulos 1989; Chandrasekhar 1990; Chandrasekhar & Ferrari 1990). Moreover, he certainly did not leave the subject of general relativity aside, as we shall see.

3. Colliding plane waves

It appears that the initial impetus that led to the next stage of Chandra's work, namely that on colliding plane waves in general relativity, was a letter from Yavuz Nutku (who had many years earlier been one of his students) which pointed out that the metric which arises when two impulsive gravitational waves with non-parallel polarization collide is described by the simplest solution of equations that Chandra had encountered in his derivation of the Kerr metric (Chandrasekhar 1978a).

There seems to be little doubt that it was Chandra's fascination with the mathematics of black holes that gradually began to turn him from a rigorous requirement that his work be directly relevant to astrophysically realistic situations. Of course, his study of black-hole perturbations was indeed astrophysically relevant, particularly because gravitational wave detectors may, before too long, possibly be able to detect the 'ringing' of a black hole after its formation or after it swallows a companion body. However, as his work continued, he became more and more seduced by the remarkable *mathematical* quality of the equations that he encountered.

Perhaps his 1989 paper with Xanthopoulos on two black holes attached to strings is the most extreme – and least 'Chandra–like' of his publications, in its departure from astrophysical realism. But colliding impulsive plane waves are also somewhat unrealistic. Since a plane wave must extend all the way to infinity, one cannot expect such a wave to be realized accurately in the physical universe. Moreover, impulsive gravitational waves (where the curvature tensor has the form of a Dirac delta function) do not provide a very reasonable idealization to the distant gravitational field of a violent event (such as, say, the congealing of two black holes) owing to the fact that there is an unending constant flux of gravitational radiation energy after the impulsive wave has passed. In addition, the nature of the space-time singularity that arises after the encounter between two such impulsive waves may have a special and unrepresentative structure, owing to its arising from a situation in which there is exact symmetry and precise focussing.

All this notwithstanding, colliding waves may well have provided an ideal framework for Chandra to make substantial progress towards an understanding of the very problem that his early researches into the equilibrium of white dwarfs had led to: the existence of *spacetime singularities.* Colliding plane waves indeed have the habit of leading to such singularities; and there is at least the possibility that these singularities may be closer to being realistic than those which occur in the Schwarzschild black hole (too special symmetry) and the Kerr black hole (closed timelike curves and intervening Cauchy horizon). It is not clear to me that Chandra was much concerned by the issue of physical realism at this stage in any case. Here was a family of solutions that he could study in detail. He could bring all of his magical gifts with equations to bear on these examples and, with luck, some deeper understanding of the nature and formation of space-time singularities could indeed come about.

His first paper on this subject described work that he and Valeria Ferrari had done (Chandrasekhar & Ferrari 1984) concerning the very situation that Nutku had described in his letter (Nutku & Halil 1977) [and which generalized the situation that Khan and I published in 1971 where the planes of polarization of the approaching impulsive plane waves were taken to be parallel (Khan & Penrose 1971)]. Their paper provided a very comprehensive analysis of this (Nutku-Halil) space time, including detailed expressions for all the NP spin-coefficients and curvature quantities.

In his next three papers on the subject, this time joint work with Xanthopoulos (Chandrasekhar & Xanthopoulos 1985a,b, 1986a) he included various matter terms into the equations: electromagnetic, perfect fluid (with $\varepsilon = p$), and null dust. Again the treatment was very detailed, but there were certain complicating issues that arose. For example, it was not reasonable to admit a delta function in the electromagnetic component to the incoming plane waves. For a delta function in the electromagnetic field would lead to a square of a delta function in the energy-momentum tensor and therefore in the Ricci curvature, which is not allowable. For there to be a delta function in the Ricci tensor, to accompany the already present delta function in the Weyl tensor describing the gravitational impulse, the Maxwell tensor would have to have the form of a square-root of a delta function. Chandra circumvented this particular problem by having the Maxwell field have only a step function at the gravitational impulse.

One source of potential confusion arose from the unusual way in which Chandra tended to obtain his solutions to the various field equations for this situation. The figure (taken from Chandrasekhar & Xanthopoulos 1986a) represents a space-time diagram for those two coordinates for which there is dynamic evolution. These are taken to be two null coordinates u and v (whose sum and difference may be regarded as the 'time' and the 'distance in the direction of propagation'), while the remaining two coordinates describe the flat plane surfaces of the waves – these planes being orbits of the two commuting translational symmetries of the space-time. Region IV is the flat spacetime between the approaching impulsive waves. Regions II and III are the portions of the two waves to the far side of the leading impulse. Region I is where the scattering between the waves takes place, this being the only region where serious work needs to be carried out in solving equations.

The flat portion of the space-time, prior to the arrival of either wave, is region IV. Chandra's ease with equations made it natural for him to start with the scattering region, and then to see what kind of waves in regions II and III might give rise to whatever solution for region I he might have found! This may seem strange to those physicists who think in terms of the evolution from Cauchy data on an initial hypersurface. In Chandra's case of the perfect fluid with $\varepsilon = p$ (the case illustrated in the figure), it turned out that in order to obtain the required solution for region I, the material in regions II and III could be taken to be null dust. The apparent difference between the equations of state holding in region I and those holding in regions II and III caused some confusion and a certain amount of controversy. Basically this was, I believe, a misunderstanding of Chandra's position concerning this situation. In fact, the perfect fluid with $\varepsilon = p$ has null dust as a limiting configuration, so there is not really any inconsistency between the different regions.

To clarify this point, the energy momentum tensor for an $\varepsilon = p$ fluid has the form

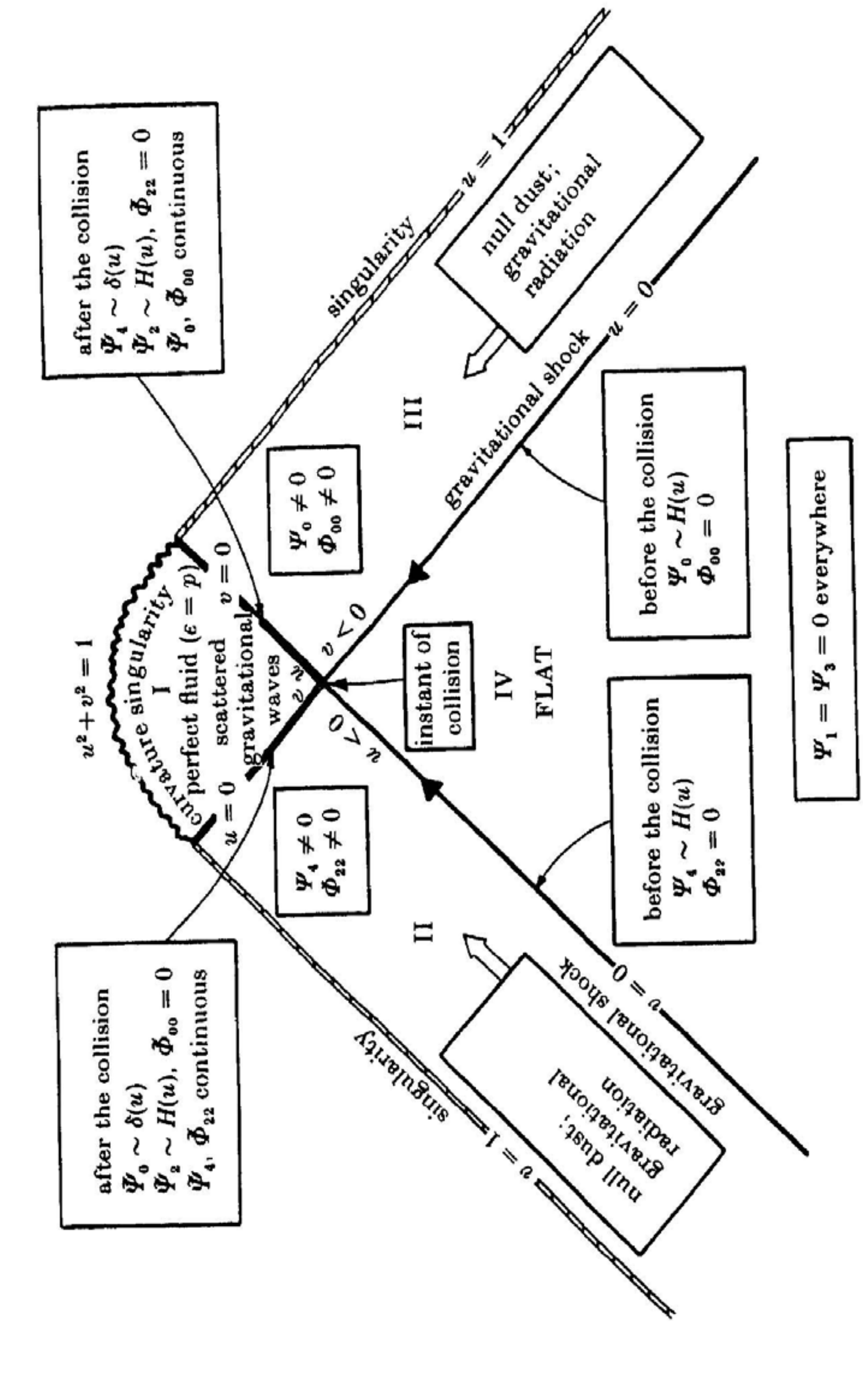
$$T^{ab} = 2v^a v^b - g^{ab}\left(g_{cd}v^c v^d\right);$$

where the vector v^a is directed along the fluid's 4-velocity and satisfies

$$v^a v_a = \varepsilon = p.$$

When v^a becomes a null vector, we get $\varepsilon = p = 0$, but this is consistent, and the energy-momentum tensor becomes $T \quad = 2v\,v$, namely null dust. Thus, null dust can be regarded as a particular (limiting) state of an $\varepsilon = p$ fluid.

In three subsequent papers (Chandrasekhar & Xanthopoulos 1986b, 1987a,b), Chandra confronted the issue of the singularity itself and the curiously slippery nature of the development of singularities in general relativity. The first of these papers presented a result which was a genuine surprise to me when Chandra first informed me about it. I had been under the impression (though without proof) that the circular line marked 'curvature singularity' at the top of the figure would always remain a region of infinite curvature for all collisions of this type. However,



after the collision
$\Psi_4 \sim \delta(u)$
$\Psi_2 \sim H(u)$, $\Phi_{22} = 0$
Ψ_0, Φ_{00} continuous
singularity
$u = 1$
null dust; gravitational radiation
$u = 0$
gravitational shock
III
before the collision
$\Psi_0 \sim H(u)$
$\Phi_{00} = 0$
$\Psi_0 \neq 0$
$\Phi_{00} \neq 0$
$u^2 + v^2 = 1$
curvature singularity
I
perfect fluid ($\epsilon = p$)
scattered gravitational waves
$v = 0$
$u = 0$
$v < 0$
$u < 0$
instant of collision
IV
FLAT
$\Psi_1 = \Psi_3 = 0$ everywhere
$\Psi_4 \neq 0$
$\Phi_{22} \neq 0$
before the collision
$\Psi_4 \sim H(u)$
$\Phi_{22} = 0$
II
after the collision
$\Psi_0 \sim \delta(u)$
$\Psi_2 \sim H(u)$, $\Phi_{00} = 0$
Ψ_4, Φ_{22} continuous
$v = 0$
gravitational shock
null dust; gravitational radiation
singularity
$v = 1$

Chandra told me of his work in progress with Xanthopoulos on a vacuum colliding wave spacetime, providing an example which, at that stage, could be seen to have non-diverging curvature there. Soon after discussing the matter with me, Chandra was able to see how to change his coordinates so as to continue the metric explicitly across that seemingly singular region, exhibiting it to have more subtle geometric character than I had imagined, with a null hypersurface appearing that plays a role similar to that of an event horizon, and beyond which lies a timelike singularity. The remaining two coordinates cannot be ignored in this discussion, and the structure of the entire space-time remains somewhat complicated. In the other two papers, analogous situations are considered in which an electromagnetic field or an $\varepsilon = p$ fluid is introduced. The curiously contrasting behaviours are examined in some detail. Chandra's work with Xanthopoulos was clearly a high point for him, and it was a particular tragedy when Xanthopoulos met his untimely death from the attack of a crazed assassin. Chandra wrote a moving tribute in his *Selected Papers*, *Volume 6* (the volume that Xanthopoulos edited). Later he played an important role in the establishment of the tri-annual Xanthopoulos prize, for work in relativity achieved by young researchers.

4. Aspects of Chandra's mathematical heritage

The papers that I have discussed above far from exhaust Chandra's voluminous research into general relativity. There are several other articles in which various aspects of exact solutions are discussed. It is striking how his involvement with general relativity seduced him more and more in the direction of mathematics, where his earlier requirements of direct relevance to astrophysics seem to have been somewhat pushed aside. It seems clear that the equations that he encountered in his work on general relativity theory gave him immense satisfaction, and he seemed continually surprised at the mysterious beauty that these equations revealed. In his book *Truth and Beauty: Aesthetics and Motivations in Science* (Chandrasekhar 1987), he remarked on this explicitly, and in his booklet entitled *The Series Paintings of Claude Monet and the Landscape of General Relativity*, his dedication lecture on the opening of the Centre for Astronomy and Astrophysics in Pune (Chandrasekhar 1992), he compared some of this mathematical beauty with a sequence of paintings by Monet. Particular instances of the vacuum equations that had impressed him so much had occurred in a comparison that he had found between the basic black-hole space-times (Schwarzschild, Kerr) and the corresponding basic colliding wave space-times (Khan-Penrose, Nutku-Halil). A similar comparison was pointed out for the Einstein-Maxwell equations. All this arose from the mathematical structure of solutions of the Ernst equation (Ernst 1968), which controls the solutions of Einstein vacuum (or Einstein-Maxwell) metrics with two commuting symmetries.

As far as I am aware, these comparisons have not yet been fully understood in terms of the various techniques that have been developed over the years to handle such space-times. The particular procedure that appeals to me the most is that developed by Woodhouse and Mason (1988), which uses the procedures of twistor theory to describe these space-times in terms of Holomorphic vector bundles over non-Hausdorff Riemann spaces. This is part of a general comprehensive treatment of integrable systems (of which space-time metrics with two commuting Killing vector provide an example), in terms of twistor theory, which these two authors have thoroughly developed in a recent book (Mason and Woodhouse 1996). In view of the many intriguing relationships with different features of integral systems that Chandra's work has thrown up, I feel sure that there is a good deal that is deep, yet to be learned, from a study of the insights that he gained from his work in general relativity.

The same can also be said of a study of his analysis of the separation of gravitational perturbations and of other systems of equations in stationary black-hole backgrounds. There is yet much mystery to be unravelled. Some of this has already been achieved in the work of Carter (1968), Walker and Penrose (1970), Carter and McLenaghan (1979), Kamran and McLenaghan (1984) and many others, whereby separation can be related to the existence of a Killing Tensor, Killing spinor and Killing-Yano tensor. There are relations to twistor theory here also, and it is my guess that a further study of Chandra's work from this direction may well throw some profound light on these issues.

In a sense, Chandra's lifetime work was like a circle, basically starting with his insights that led us to believe that too massive white dwarfs must collapse to a space-time singularity, and finally reaching back to a sophisticated study of those very singularities. Yet, for all Chandra's extraordinary ingenuity and industry, the deep answers are still missing. I rather believe that Chandra did not really expect that this work might directly find an answer, however. The main thrust of his work in this area was somewhat different. It was driven more and more by the quest for mathematical elegance, coupled with a deep belief in a profound underlying connection between physics and mathematics. Perhaps it is here that the answers are finally to be found.

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