The Virial Method and the Classical Ellipsoids

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1. Introduction

Chandra completed his book *Hydrodynamic and Hydromagnetic Stability (HHS)* in 1960 and, as was his custom, turned his attention to a new area of research. He began to study general relativity at this time, and it appeared that this would be his exclusive new direction. However, as the result of a pair of accidents, Chandra in fact devoted much of the period from 1960 through 1968 to the virial method and to an analysis of the figures of the classical ellipsoids and their stability. This subject and the general theory of relativity competed for his attention during these years. It was only after completion of his book *Ellipsoidal Figures of Equilibrium (EFE)* (Chandrasekhar 1969) in 1968 that he felt able to devote himself primarily to the subject of relativity, which then was the principal occupation of the remainder of his research career. His enthusiasm for the development of the classical ellipsoids waxed and waned during this period, and he wrote that parts of it were performed ‘under protest,’ his sense of responsibility to the subject taking precedence over his inclination to enter more fully into the study of relativity.

The virial theorem, in scalar form, has a history in astronomy (cf. Ambartsumyan 1958). In the theory of stellar pulsations, it was employed by Ledoux (Ledoux 1945) to obtain an approximate expression for the lowest mode of radial pulsation and the effect on that mode of a slow rotation. Tensor forms of the virial theorem had been employed by Rayleigh (Rayleigh 1903) and by Parker (Parker 1957) in special contexts, and Chandra had long had in the back of his mind the notion that one could use this form of the theorem to obtain useful, approximate information about figures seriously distorted from the spherical by rotation or magnetic fields. He had included the basic equations in his preparation of *HHS* with this in mind. I was at this time one of his research students and it was therefore natural that he suggest to me, as part of my dissertation, the development of this method. The application he proposed was the problem of the oscillations and stability of the Maclaurin spheroids.

This would represent a test case: the frequencies of the Maclaurin spheroids were known, and the virial equations, along with a linear ansatz for the Lagrangian displacement as in Ledoux's problem, would lead to approximate frequencies which could then be compared with the exact values. The ansatz would be needed because the virial method is a moment method which would require some kind of approximate closure procedure, such as that provided by the ansatz. What neither of us
anticipated was the discovery, in winter 1960–61, that the virial equations, in the context of these incompressible figures, form a closed system and therefore give the exact frequencies, without the need for an ansatz (Lebovitz 1961). This was the first accident diverting Chandra’s attention from relativity to the virial theorem and the classical ellipsoids: although he had hoped the virial method would be powerful, he now realized that it was more powerful than he had expected it to be. It presented an elementary alternative to the analysis via expansions in ellipsoidal harmonics, which was the method employed in the lengthy and arduous analyses of the Jacobi ellipsoids carried out in the latter part of the nineteenth century and the early part of the twentieth century by Poincare, Darwin, Lyapunov and Jeans. This new method should allow one to simplify and extend these analyses. This powerful technique needed to be developed more fully. And this he began to do in earnest, as described below, developing the machinery needed to apply the virial method to rotating, self-gravitating masses and applying this machinery to study the linearized stability both of the Maclaurin and Jacobi sequences (to verify and extend the classical analyses) and of compressible, rotating masses as well.

This program had come to a stage of apparent completion in the spring of 1964, and a summarizing paper (Chandrasekhar & Lebovitz 1964) had even been written. Chandra was invited at this time to speak at the Courant Institute in New York City when, browsing during an hour of leisure in Stechert’s bookstore, he chanced upon a copy of Bassett’s *Hydrodynamics* (Bassett 1888), and purchased it. This was the second of the pair of accidents. Bassett’s book contains an account of the Riemann ellipsoids, discovered and discussed by Dirichlet, Dedekind and Riemann in the period 1857–1861 – long before the work of Poincaré and others on the Jacobi ellipsoids, but barely alluded to in their work, and unknown to Chandra until he looked into Bassett’s book. The Riemann ellipsoids represent a substantially more general family of solutions of the equations of the fluid dynamics of self-gravitating figures than those presented by the Maclaurin and Jacobi families, and their properties had been much less fully explored. On the one hand, one’s understanding of the possible figures of self-gravitating masses and their stability was evidently much narrower than it could be – and should be – and, on the other hand, he now had sufficient technique to bring the understanding of these more general ellipsoids to the point of development that had been achieved for the Jacobi and Maclaurin families. He resolved to bring this beautiful but neglected theory to a fuller stage of completion.

In 1963, Chandra had given the Silliman lectures at Yale University on the subject of ‘Them Rotation of Astronomical Bodies’. These lectures were to be written up in book form. Chandra put off doing this during the period when the study of the Riemann ellipsoids was taking place and subsequently used this opportunity to expand the lectures into his book *EFE*, which encompassed his own research and that of his students and collaborators over the period in question. On its completion in 1968, his formal association with the subject ended.

Chandra was deeply interested in scientific and intellectual history and in the motivations of successful scientists, scholars and artists. He admired the funeral
essays given by serious scientists of the past on the subject of colleagues who had recently died. Those that he admired most were not eulogies but rather analyses of the contributions of the scientific personality who had recently died by someone able to place those contributions in a general scientific perspective. Indeed Chandra himself was impatient with fulsome but vague praise of his own work and preferred constructive criticism based on an understanding of the subject. The practice of funeral essays, a feature of a more leisurely era, has lapsed, and it is in any case difficult to imagine any one individual able to place Chandra’s diverse contributions into perspective. The current volume, however, may indeed serve the kind of purpose Chandra would have admired and respected.

Over the years, from time to time, he wrote a chronology of his research efforts during a certain space of time, together with remarks on the scientific and personal background for this period of his research. A number of these he copied and sent to me. I have benefited in writing the current article from his own observations for the period from 1960 to 1968.

Sections 2 and 3 below present background to the subject matter: a brief history of the classical ellipsoids and an explanation of the virial method with its advantages and disadvantages. Section 4 provides a description of some of Chandra’s contributions during this period (roughly 1960–1968) and, finally, section 5 provides a retrospective view including a sampling of subsequent developments. I have not attempted to include a systematic bibliography, since this can be found in EFE.

2. A History of the ellipsoidal figures

Isolated discoveries regarding the ellipsoidal figures, like those of Maclaurin and Jacobi, occurred over a long period of time. Beyond these sporadic events we may identify two principal periods of the development of the theory of the ellipsoidal figures and their stability. The first of these occurred in the middle of the last century and was initiated by Dirichlet, and the second began toward the end of that century and was initiated by Poincare and Lyapunov.

2.1 Dirichlet’s problem

In his lectures on partial differential equations for the term 1856–7, Dirichlet included a description of certain solutions he had found of the equations of inviscid fluid dynamics. He had begun to write these up into a coherent whole, but his untimely death prevented the completion of this project. The completion was left to Dedekind, who not only put together the completed sections that Dirichlet had left, but also organized scattered notes into further sections of the paper, and followed

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1The monograph by Lyttleton (Lyttleton 1953) contains a more extensive historical development of these figures, including an account of the role they played in the fission theory.
the completion of Dirichlet’s paper with a paper of his own, containing what is now called Dedekind’s theorem. These papers (Dirichlet 1860; Dedekind 1860) are published consecutively in the same issue of the Journal für die Reine und Angewandte Mathematik of 1860. Riemann’s paper (Riemann 1861), published the following year in the Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, reorganizes Dirichlet’s solution into a somewhat different form (closer to that found in EFE), and discusses the steady state solutions and their stability to disturbances leaving them ellipsoidal. Subsequent research followed Riemann’s presentation and these ellipsoidal solutions (especially the steady-state solutions) are now usually referred to as the Riemann ellipsoids. However, the important mathematical observations underlying the existence of these solutions are due to Dirichlet, as Riemann himself emphasizes. These observations are two.

The first is that a velocity field that is a linear expression in the cartesian coordinates ‘linearizes’ the equations of fluid dynamics in a sense described below. Dirichlet made a point of using the Lagrangian form of the fluid-dynamical equations to introduce this form of the velocity field. In the more familiar Eulerian form of the equations of fluid dynamics, the effect of this assumption is that the nonlinear advective term then also becomes a linear expression in the cartesian coordinates. It is in this sense that Dirichlet’s assumption linearizes the equations: if there were no nonlinear forcing terms present, the equations of fluid dynamics (equations 2 and 3 below) would become linear in the cartesian coordinates, and one could immediately solve them to obtain a finite system of ordinary differential equations. The idea of linearizing the equations of fluid dynamics through such an assumption has been rediscovered repeatedly (e.g., Craik 1989). The second observation is that the self-gravitational force inside an ellipsoid of uniform density is also given by a linear expression in the cartesian coordinates. Putting these observations together, and taking due notice of the conditions at the free boundary and the assumption of incompressibility, Dirichlet was led to a system of ordinary differential equations governing the parameters of the system (the semiaxes $a_1$, $a_2$, $a_3$ of the ellipsoid, and six parameters characterizing the velocity field). While the velocity field is linear in the cartesian coordinates, the differential equations governing the parameters are nonlinear. The Maclaurin and Jacobi families, which are in equilibrium in a rotating reference frame, form a small subclass of solutions of this system.

Dirichlet’s problem thus provides a physically meaningful context wherein a daunting system of partial differential equations is reduced to a system of ordinary differential equations of finite order (in the general case, of order twelve). Dirichlet had applied his equations already in 1857, to the following problem (Dirichlet 1860). For the Maclaurin spheroids, it had been observed earlier that there is a maximum angular velocity. That is, if the density is prescribed and one considers a sequence of Maclaurin spheroids of increasing angular momentum (and therefore

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2 Dirichlet’s argument for preferring the Lagrangian form in his context makes very interesting reading.
of increasing eccentricity of meridional cross-section), the angular velocity of the figure at first increases, but subsequently decreases, a maximum occurring at a certain critical value of the angular momentum (cf. Chandrasekhar 1969, p.79). On the other hand, one can consider a spheroidal fluid mass of the prescribed density whose initial velocity is that of pure rotation with an angular speed exceeding that which is possible for a Maclaurin spheroid. What is the dynamical outcome of these initial data? This question can be addressed in the context of Dirichlet’s equations (with the result that the object performs an oscillatory motion).

In the course of editing Dirichlet’s notes for publication, Dedekind observed a certain reciprocity in the system of equations, which can be explained in the following way. The fluid velocity consists of two parts: an angular velocity of rigid-body rotation, and a motion of uniform vorticity superimposed on the latter. Each of these motions can be characterized by a three-component vector (time-dependent, in the general case). Interchanging these vectors provides a different solution of the equations for which the geometric figure is the same (i.e., the semiaxes of the ellipsoid are identical in the two motions). In EFE these two motions are said to be adjoint to each other, since they are obtained by taking the transpose, or adjoint, of a certain matrix. An example of such a pair of adjoint configurations is the Jacobi-Dedekind pair: the Jacobi ellipsoid is at rest in frame of reference rotating about the $z$-axis with angular speed $\omega$ and the Dedekind ellipsoid is at rest in the inertial frame but with a fluid velocity of constant vorticity $\zeta = -\omega (a_1^2 + a_2^2)/a_1 a_2$.

Riemann rederived Dirichlet’s equations in a more symmetrical form. His derivation of Dedekind’s reciprocity law consists of a single remark. He went further, however, than merely giving a more compact formulation than his predecessors. He also considered quite generally the family of equilibrium solutions of the system of ordinary differential equations (which correspond to steady-state solutions of the Euler equations of fluid dynamics). These he found to be divided into two kinds: those for which the angular velocity and vorticity are aligned along a principal axis of the ellipsoid, and those for which the latter is not true but these vectors lie in a principal plane. He then used the system of ordinary differential equations to study the stability of these steady-state solutions to disturbances of the fluid mass leaving it an ellipsoid. For this he employed a variation of Lagrange’s minimum-energy method. The parameter space is two-dimensional: the ratios of semiaxes, say $\alpha = a_2/a_1$ and $\beta = a_3/a_1$, may be chosen as parameters. Then the part of the parameter space occupied by steady-state solutions of the kind considered is a certain region in the $\alpha \beta$-plane. Riemann’s method led to the identification of critical points, or critical curves, separating stable from unstable subregions of this region of parameter space.

He further noted that it was feasible to generalize the stability theory by sub-

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3 The term equilibrium is somewhat ambiguous. Equilibrium solutions of Dirichlet’s system of ordinary differential equations correspond to steady-state solutions of the Euler equations of fluid dynamics which, with few exceptions, are not true equilibria, or even relative equilibria, of that system. We will live with this ambiguity, as in the title of Chandra’s book EFE.
jecting the Euler equations to initial data representing arbitrary (hence in general non-ellipsoidal) disturbances of the steady-state solutions, since doing so would lead to linear partial-differential equations, although he did not pursue this himself.

2.2 The fission theory

The occurrence of multiple systems of objects in the sky is the most obvious feature of the solar system, but is by no means limited to the solar system. For example, about half the stars in the sky are double stars. How these and other multiple systems form is a continuing issue of current research. A clear statement of an idealized mathematical problem bearing on this issue appears in the classic dynamics text by Thomson and Tait (1879). The underlying idea is that a rotating, self-gravitating fluid mass, initially symmetric about the axis of rotation (like a Maclaurin spheroid), can undergo an axisymmetric evolution in which it first loses stability to a nonaxisymmetric disturbance, and continues for a while evolving along a non-axisymmetric family (like the Jacobi family) toward greater departure from axial symmetry; then it undergoes a further loss of stability to a disturbance tending toward splitting into two. These authors made various plausible conjectures regarding this fission theory in the context of the known, rigidly rotating figures of Maclaurin and Jacobi.

The problem of fleshing out the mathematical skeleton constructed by Thomson and Tait was taken up independently by Lyapunov (Lyapunov 1884) and by Poincaré (Poincaré 1885). Their mathematical treatment of this problem went beyond the particulars of the astronomical problem and laid the groundwork for the area of nonlinear analysis known today as bifurcation theory. In the context of the rigidly rotating figures of Maclaurin and Jacobi, the most relevant perturbations of these figures appeared to be those associated with deformations of the free surface described via ellipsoidal harmonics of orders two and three. Ellipsoidal harmonics of order two are, in the limit of linear disturbances, of the kind envisaged by Riemann: the disturbed figure remains an ellipsoid. Ellipsoidal harmonics of order three or higher are not of this kind, and the corresponding analysis carried out by Poincaré and Lyapunov is significantly more complicated for disturbances of this kind.

The outcome of these mathematical analyses did not fully confirm either the speculations of Thomson and Tait or the further speculations of Poincaré, and despite subsequent efforts and clarifications by Jeans (1917), Cartan (1928) and others, the issue of the viability of the fission theory remains unsettled to this day. From the standpoint of the mathematical analysis of the classical ellipsoids, the advances consisted of determining the stability of the Maclaurin and Jacobi figures to certain higher-harmonic disturbances: arbitrarily high in the case of the Maclaurin figures (Bryan 1889; Cartan 1928), through fourth harmonics in the case of the Jacobi figures (Appell 1921).

Thus Riemann’s remark, that one could determine the stability of the more general class of ellipsoids discovered by Dirichlet, to arbitrary disturbances, remained only partially explored even for the small subclass of figures represented by the
Maclaurin and Jacobi figures, and essentially unexplored where the more general Riemann ellipsoids were concerned.

This was the state of the subject at the time when Chandra ran across Bassett’s account of it.

3. The virial method

The virial theorem has long been used in mechanics to obtain estimates of dynamical motions of systems of particles. It is obtained by taking the scalar product of either side of the force-balance equation with the position vector of the $j$th particle and summing:

$$\sum_{j=1}^{n} m_j x_j \cdot \frac{d^2 x_j}{dt^2} = \sum_{j=1}^{n} x_j \cdot f_j (x),$$

(1)

where $n$ is the number of particles. After some elementary manipulations and (possibly) the introduction of plausible assumptions, it provides a relation between the inertial terms on the left and the forcing term on the right. It is known to astronomers in particular for estimating the relative importance of gravitational and inertial effects in groups of stars (as in Ambartsumyan’s book [1958]).

Its application in fluid dynamics was considered by Rayleigh (1903) and more recently by Parker (1957). To carry out this application in the case of an ideal fluid, one considers the force-balance equation (the Euler equation)

$$\frac{Dv_i}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + f_i (x, t), \quad i = 1, 2, 3.$$  

(2)

Here $v_i = v_i (x, t)$ is the $i$-th component of the velocity, $p$ the pressure, and $f$ the force per unit mass. The operator $D/Dt$ is given by the formula

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial}{\partial x_i}.$$  

These equations have to be supplemented by others to form a closed system of equations. For the sake of definiteness we suppose the fluid to be incompressible: $p = \text{constant}$. This imposes the further condition that

$$\text{div} \, v = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0,$$  

(3)

and the system of equations (2) and (3) is then closed. It still needs to be supplemented by appropriate initial and boundary conditions. If one multiplies the $i$-th equation of the system (2) by $x_i$, sums on $i$ from 1 to 3, and integrates over the domain $D$ occupied by fluid, one obtains the analog of the virial theorem for a collection of particles. This clearly results in a relation among integrals involving inertial terms (from the left-hand side of the equation) and terms involving forces,
due to fluid pressure and whatever further forcing terms are present (from the right-hand side of the equations). This relation is called the virial equation (or scalar virial equation).

It is not a priori evident that the relation obtained in this way will be a useful one. However, if you grant that it may be useful, there are immediate generalizations of it that may then also be useful. Instead of multiplying the \( i \)-th equation by \( x_i \) and summing, one can multiply the \( j \)-th equation by \( x_i \) and have two free indices, \( i, j \) providing nine equations in all:

\[
\int_D x_i \frac{Dv_i}{Dt} dV = -\int_D x_i \frac{\partial p}{\partial x_j} dV + \int_D \rho x_i f_j (x, t) dV, \quad i, j = 1, 2, 3
\]

or, after manipulating the left-hand side,

\[
\frac{d}{dt} \int_D x_i v_j dV - \int_D v_i v_j \rho dV = -\int_D x_i \frac{\partial p}{\partial x_j} dV + \int_D \rho x_i f_j (x, t) dV, \quad i, j = 1, 2, 3.
\]

The condition (3) must also be taken into account (even in the scalar case). For an ellipsoid of semi-axes \( a_1, a_2, a_3 \) it implies the further relation

\[
\int_D \left( \frac{x_1 v_1}{a_1^2} + \frac{x_2 v_2}{a_2^2} + \frac{x_3 v_3}{a_3^2} \right) dV = 0.
\]

These equations must contain more information than the earlier scalar virial equation since the latter is derivable from them. They form the so-called tensor virial equations. One need not stop there: the \( k \)-th equation of the system (2) can be multiplied by \( x_i x_j \) and integrated over \( D \), providing twenty-seven equations in all. And so on.

A paradigm for investigating differential equations describing the evolution of an interesting physical system is: (a) find the steady-state solutions, (b) investigate their stability. Ledoux’s application, and the initial investigations employing the virial tensor in the context of the ellipsoids, was in the context of stability of known steady-state figures. For this application of the tensor virial equations one needs these equations, not in the form given by equation (5), but in a form derived from the latter by perturbation about a known solution. In other words, one considers equation (5) for the known solution, the same equation but for the unknown (or perturbed) solution, and subtracts. The difference may be written conveniently in term of the Lagrangian displacement \( \xi(x, t) \). The latter is the vector from the position \( x \) of a fluid particle in the unperturbed flow to the position of the same fluid particle for the perturbed flow. A knowledge of the Lagrangian displacement as a function of position and time provides a complete description of all the flow variables in the present, conservative context. If we define the variables

\[
V_{i;j} = \int_D \rho \xi_j x_i dV \quad \text{and} \quad V_{ij} = V_{i;j} + V_{j;i},
\]
we can express the tensor virial equations as follows (see EFE for details):

\[
\frac{d^2}{dt^2}V_{ij} + \frac{d}{dt} \int_V \rho \left( \xi_j v_i - \xi_i v_j \right) dV = \delta T_{ij} + \delta W_{ij} + \omega^2 V_{ij} - \omega_i \omega_j V_{kj} \\
+ \delta_{ij} \delta \Pi + 2 \epsilon_{ilm} \omega_m \delta \int_D \rho v_i x_j dV, \tag{7}
\]

where \( \delta \) indicates the difference between the perturbed and unperturbed version of the expression following it. Here double indices not separated by a semicolon are symmetric in their indices. The terms in \( \delta T, \delta W \) and \( \delta \Pi \) refer to quantities involving kinetic energy, potential energy and pressure respectively. The first two of these can be expressed in terms of the variables \( V_i \). The incompressibility condition can be shown to imply that

\[
\frac{V_{11}}{a_1^2} + \frac{V_{22}}{a_2^2} + \frac{V_{33}}{a_3^2} = 0. \tag{8}
\]

For an incompressible ellipsoid, these equations, nine in number after the term involving the pressure is eliminated, represent a homogenous system in the nine unknowns. Thus no ansatz is required. At least some of these nine quantities are non-zero if the surface deformation of the ellipsoid is given by a second-order ellipsoidal harmonic. Hence, among the solutions for the oscillation frequencies are those belonging to the second-harmonic perturbations of the ellipsoid. To achieve perturbations of the ellipsoid given by third harmonics, one requires the higher-order virial equations obtained by taking moments of the \( i \)th equation with \( x_i x_k \).

And so on.

The advantages of these equations in comparison with methods employed earlier, involving expansions in ellipsoidal harmonics, may be explained as follows. The use of the latter is arduous in part because the ellipsoidal coordinates suffer from a feature not shared by the more widely used coordinate systems (like Cartesian and spherical coordinates): they are not a single coordinate system but a parameterized family of coordinate systems depending on the semi-axes of the particular ellipsoid. Correspondingly, the ellipsoidal harmonics form not a single complete system of functions but a parameterized family of complete systems: they have to be calculated anew for each ellipsoid under investigation. In using expansions in ellipsoidal harmonics, one uses (explicitly or implicitly) the orthogonality properties of these functions when integrated over the fundamental domain to project a function onto some finite-dimensional subspace. Likewise with spherical harmonics. For expansions in cartesian coordinates, the virial method turns out to play the role of the projection procedure. It has the advantage that the ‘harmonics,’ which are simply the monomials in powers of the cartesian coordinates, are known once and for all and do not depend on the object under investigation. It has the disadvantage as well that it is necessary to work out a different set of virial equations corresponding to each order of ellipsoidal harmonics. Equation (7) above corresponds to second-order harmonics. The system that would be obtained by multiplying the
kth equation by the arbitrary monomial $x_i x_j$ corresponds to third-order harmonics. These systems rapidly become unwieldy and are limited for most practical purposes to low orders.

It is also true in the context of the incompressible ellipsoids that the equations of the unperturbed flow are given by the tensor virial equations (5), in the following sense. If one substitutes into these equations the structure of Dirichlet’s solution with unknown parameters, the tensor virial equations then determine the relations among the parameters.

There is no link in principle between the virial equations on the one hand and the classical ellipsoids on the other. The virial equations can be formulated for arbitrary kinds of fluid configurations, with corresponding changes in the forms they take. They are, however, particularly well adapted to the study of the ellipsoids.

4. Chandra’s contributions

The work of Chandra and his collaborators broke ‘old ground’ as well as ‘new ground.’ The old ground consisted of applications of the virial method to problems concerning the Maclaurin and Jacobi families that had been considered earlier, especially within the context of the fission theory.4 The new ground consisted of novel applications, not only to the Riemann ellipsoids but also to other fluid-dynamical problems. Some of these contributions are now summarized.

4.1 Old ground

The early applications were to the Maclaurin and Jacobi figures (recall that Chandra did not know of the more general figures of Riemann until 1964). The oscillation frequencies of the Maclaurin figures were calculated for perturbations associated with second- and third-order ellipsoidal harmonics. The location of bifurcation points under surface deformations described by third-harmonics along the Maclaurin and Jacobi families was also carried out using the virial equations. Since the location of these points were either already known or deducible on the basis of already established technique, one may ask why one should do them again. There are complementary reasons for this.

Recalculating the classical bifurcation points from the new standpoint reconfirms the older results from a computationally distinct viewpoint, validates the new procedure and provides computational experience with the new technique that will be needed in breaking new ground. There had been ample confusion regarding the interpretation of the classically calculated bifurcation points (cf. Lyttleton 1953, chapter 1 regarding this), and therefore scope for reconfirmation. Computational experience with the new technique was important in order to take advantage of the

4 Chandra chose not to address the fission theory directly. To do so would have involved a heavy investment in nonlinear bifurcation analyses, whereas he was more interested in exploring the capabilities of the method in linear theory.
virial method. This preliminary series of investigations showed very convincingly that one could indeed find all the critical points that had been found classically with an essentially elementary technique, i.e., without ever constructing, or even explicitly introducing, the ellipsoidal harmonics.

4.2 New ground

The first novel applications of the virial method were to compressible, rather than incompressible masses. For this application a variant of equation (7) is needed. These equations required a special development to handle the gravitational terms, leading to the *superpotentials*, scalar quantities generalizing the gravitational potential. These developmental matters were attended to in a series of papers (cf. Chandrasekhar & Lebovitz 1962a, b). The equations could then be applied in specific contexts (including rotating polytropes, for example; cf. Chandrasekhar & Lebovitz 1962c). An application of the results to a concrete astronomical problem was the interpretation of the beat period of the \( \beta \) Canis Majoris stars (Chandrasekhar & Lebovitz 1962d, e, f). The compressible theory for a spherical star indicated that, for a critical value \( \gamma = 1.6 \) of the ratio of specific heats, the fundamental mode of radial pulsation and the \( P_2 \) mode of nonradial pulsation were degenerate (i.e., have the same frequency). A small rotation would lift the degeneracy, and neither of the two resulting normal modes was radially symmetric. The result is that, under the influence of an essentially spherical forcing, both modes would be excited with comparable amplitudes, resulting in a steady beating with a frequency given by the difference of the two characteristic frequencies.5

The rediscovery of the Riemann ellipsoids opened extensive new ground for the application of the method. However, the first step was again intended to be ‘old ground’: Riemann, in his paper of 1861, had discussed the stability of the equilibrium solutions that he had found to perturbations leaving them ellipsoidal. This made it possible to consider stability in the context of the ordinary differential equations describing the ellipsoidal motion. He described his conclusions by giving the neutral curves in the space of the parameters \( \alpha a_2/a_1 \) and \( \beta = a_3/a_1 \); these represent figures on the borderline of instability, separating stable subregions of the parameter space from unstable subregions. Chandra set out to confirm this with the aid of the tensor virial equation (7). What he found for stability boundaries agreed in some domains of parameter space, but showed discrepancies in others. The pattern of discrepancy was such that, wherever Riemann concluded stability Chandra agreed, but there were small regions where Riemann concluded instability but Chandra concluded stability. The natural inclination to defer to the great German mathematician conflicted with a careful re-examination of both his methods and Riemann’s. Riemann did not calculate sets of oscillation frequencies, but rather used a version of Lagrange’s theorem: he found a function constant

5 This interpretation of the beat phenomenon was received coolly by the larger community of astronomers.
on orbits, the vanishing of whose gradient gives the equilibrium conditions, and associated stability with minima of this function. However, the converse association of instability with critical points that fail to be minima, was not justified. In Lagrange’s theorem the latter association is justified because the conserved quantity is the sum of a positive-definite kinetic energy and a potential energy. Riemann’s conserved quantity takes this form only for a subfamily of his ellipsoids (the S-type ellipsoids, defined below) and here his stability conclusion is in exact agreement with that of Chandra’s virial analysis. The pattern of discrepancy is consistent with a mis-application of Lagrange’s theorem (Lebovitz 1966). Hence what was to have been old ground opened new ground instead, correcting aspects of Riemann’s analysis.

The problem of the oscillations and the stability of the Jacobi family is of particular significance since the point along that family where instability sets in played a major role in the fission theory. Cartan (1928) had shown that the Jacobi family becomes dynamically unstable at the point of bifurcation along this family originally isolated by Poincare (1885), from which the pear-shaped family bifurcates. He had not, however, explicitly calculated the oscillation frequencies of the Jacobi family under perturbation by the associated third-harmonics. These frequencies, found in detail via the virial technique (Shore 1963), confirm Cartan’s theorem in a graphic manner.

The S-type ellipsoids, already referred to above, are a subfamily of the Riemann ellipsoids for which the angular velocity and vorticity are directed along the same line (the z-axis, say). Chandra also considered their stability to third-harmonics disturbances, but only for neutral disturbances (i.e., oscillation frequencies were not calculated). This enabled a generalization to this family of ellipsoids of the analysis of Poincare for the Jacobi family, in keeping with the intention of bringing the study of the Riemann ellipsoids to the level of completion that had previously been achieved for the Jacobi family.

Another important mathematical element of the fission theory of binary stars was the assertion, by Thomson and Tait (1879), that the Maclaurin spheroids would become (secularly) unstable to an ellipsoidal disturbance at the point where the Jacobi family bifurcates from it if dissipation is present, and not otherwise, i.e., that dynamical instability does not set in at this point. The latter point had been explicitly demonstrated, but the former had not. While the reasons given by Thomson and Tait were sound and generally accepted, an explicit confirmation was presented only in the 1960’s (by Roberts and Stewartson [1963] and by Rosenkilde [1967]). Rosenkilde’s approach was to use the virial theorem for a viscous liquid with an ansatz for the Lagrangian displacement drawn from the inviscid theory. The two approaches give the same result, fully confirming the assertion of Thomson and Tait. But Rosenkilde’s approach is remarkable for its simplicity.

Chandra also considered problems in which tidal forces join with rotation to determine the shape of the free surface, with the approximation of the tidal force such that the figures are ellipsoids. One of these, the Roche problem, envisages a liquid figure tidally distorted by a point mass. Here again the issue of secular
stability arises, and again was settled in a remarkably simple fashion by Rosenkilde’s method.

As I have mentioned, Chandra resented some of the time spent on the ellipsoidal figures because of his eagerness to continue his work in relativity. His experience with the ellipsoids served him well, however, in his subsequent research on gravitational radiation in the post-Newtonian approximation. Here he found (Chandrasekhar 1970) a useful paradigm in the adjoint Jacobi and Dedekind figures, the first radiating because its figure is rotating in an inertial frame, the second not radiating because its figure is at rest in an inertial frame. A now-standard and widely quoted reference on dissipation through the effects of gravitational radiation and of viscosity is the work of Detwiler and Lindblom (1977), which takes as its point of departure the theory of the S-type Riemann ellipsoids.

5. A retrospective view

During the period when he worked on the classical ellipsoids, Chandra endured criticism from a number of astronomers, many of whom felt that the ellipsoids were not relevant to the mainstream problems of astronomy. Chandra himself expressed impatience with the subject from time to time. One can therefore fairly inquire what the outcome of this intense research activity has been, what lasting influence it has had, in astronomy in particular and in science more generally. Indeed he addressed these questions himself in an epilogue to EFE. There he limits himself to two remarks: (1) this physically motivated and mathematically beautiful subject had been badly neglected, and it seemed a pity to leave it in such a neglected condition, and (2) he wanted to give a substantial exposition of the virial method, which has applications beyond the classical ellipsoids. I would add a third goal, however, which he expressed personally, regarding the difficult and time-consuming procedure of writing EFE: he felt that, if he did not make this effort, the classical ellipsoids and the preceding efforts over a period of almost nine years would largely be forgotten. Now, some three decades later, we can perhaps address the success of these goals.

Regarding the popularization of the virial method, success has been modest. There indeed has been a stream of applications over a long period, and the stream, while never a roaring current, does not seem to be dying out. The form of the virial theorem found in current textbooks (cf. Binney & Tremaine 1987; Shore 1992) is the tensor form. However, its extensive application supplanting expansions in harmonics has not caught on. There are reasons for this. One is that it has the disadvantage, mentioned above, that a different set of virial equations has to be defined at each order, and these become increasingly cumbersome at higher order. Furthermore, many users of EFE refer to Chandra’s technique as ‘sophisticated.’

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6 This attitude of his colleagues was not restricted to this period of his work: Chandra has said that he was never part of the astronomical establishment.
He intended exactly the opposite! However, the technique, while elementary in the sense of not requiring a knowledge of ellipsoidal harmonics, nevertheless requires its own specialized development, which is not part of a standard scientific education.

There is, however, one particular success of the method that has never been followed up and fully explained. This is what I have called ‘Rosenkilde’s method,” for flows of low Reynolds number (i.e., small viscosity). The standard approach to estimating the effect of a small viscosity is boundary-layer theory, and it can lead to very heavy calculations (cf. Roberts & Stewartson 1963), while Rosenkilde’s method is extremely simple and elegant by comparison. Its success must be related to the circumstance that the underlying equations incorporate the exact, viscous boundary conditions while the ansatz introduces the ‘outer solution’ of boundary-layer theory. However, neither the details of this correspondence nor the limits of the method’s validity have been adequately explored.

Regarding the neglected state of the subject of the Riemann ellipsoids, Chandra’s efforts clearly made significant restoration. The stability to second harmonics was considered ab initio and Riemann’s conclusions corrected. Bifurcation points under third-harmonic disturbances were worked out for the S-type ellipsoids. There are isolated cases where the same is done under fourth-harmonic disturbances. This brought the subject to a similar level of completion to that which had previously existed for the Maclaurin and Jacobi figures. However, a complete study of the dynamics of the Riemann ellipsoids was not, and has not yet been, achieved (although further progress has in fact been made recently; see below). Riemann’s remark, that the study of the stability of these figures leads ‘only to linear differential equations,’ now sounds rather innocent in view of the effort needed to make progress in the subject.

The goal of writing the book, to prevent the subject of the Riemann ellipsoids and the advances Chandra and his collaborators had made in it from disappearing from the scientific scene, has succeeded admirably. That EFE has become the principal reference on the classical ellipsoids is of course true, but this statement doesn’t go very far since EFE is the only extensive reference. It is, however, further true that the book has brought these figures to the attention of astronomers, physicists and mathematicians (to name those areas in which I have personal knowledge of research activity), allowing applications to be made that might not have occurred to their authors if there had been no such book. Many of the applications in astronomy (where real stars do not conform to the rigid hypotheses of the theory of the ellipsoids) and in physics (where the liquid-drop model of the nucleus involves figures only approximately ellipsoidal and involving Coulomb forces and surface tension rather than gravity) have an approximate character. For mathematicians, the Riemann-Dirichlet equations represent a rich Hamiltonian system harboring a variety of behaviors. For all of these, EFE is a well-known, well-written and easily accessible guide to the subject.

Chandra’s pattern of writing a book and moving to a new subject has sometimes intimidated those who wished to work in the field he just left: there is concern that everything worth doing has been done. His reason for establishing this pattern was
quite different: he wanted to state what he had learned of the subject in a coherent form. This should be a help to those who wish to study the subject further rather than a hindrance, and indeed it has been. I’ll conclude with three recent examples of research activity extending our understanding of this area of science which Chandra resurrected.

One area that was clearly not exhausted by Chandra is that of the stability of the Riemann ellipsoids. One recent development has been a reconsideration of the stability of the S-type ellipsoids (not via the virial method, but with the aid of the ellipsoidal harmonics and some help with symbol-manipulation computer programs). Oscillation frequencies have been calculated for disturbances up to fifth harmonics, and have been complemented by a WKB analysis for arbitrarily small wavelengths (Lebovitz & Lifschitz 1996a, b). These reveal fluid-dynamical instabilities associated with the strain component of the velocity field (rather than with the energetics associated with the gravitational and rotational fields). These previously undetected instabilities affect most of the parameter space, and have rather large growth rates for the Dedekind family and nearby figures, which are characterized by large strain. This is not the place to speculate on the implications these new results have for the applications of these classical figures.

Another recent development is the discovery (Marshalek 1996) of a limiting form of Riemann ellipsoids not of type S. This is an irrotational family of figures whose angular velocity does not lie along an axis but in a principal plane, overlooked by Riemann and not pointed out in EFE. It has similarities with the tilted rotor model of recently discovered atomic nuclei.

Finally, in Darwin’s tidal problem, it has been pointed out (Lai et al. 1994) that if, instead of using an approximation to the tidal potential making the figure exactly an ellipsoid, one uses a variational principle in which the linear velocity field appears in the form of a trial function, the restriction of Darwin’s tidal problem to congruent masses can be relaxed, and an improved formula for the angular velocity be obtained, consistent with arbitrary masses for the two components.

References


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7 Indeed, the stability conclusions for these figures, even for third-harmonics disturbances as in EFE, require revision.